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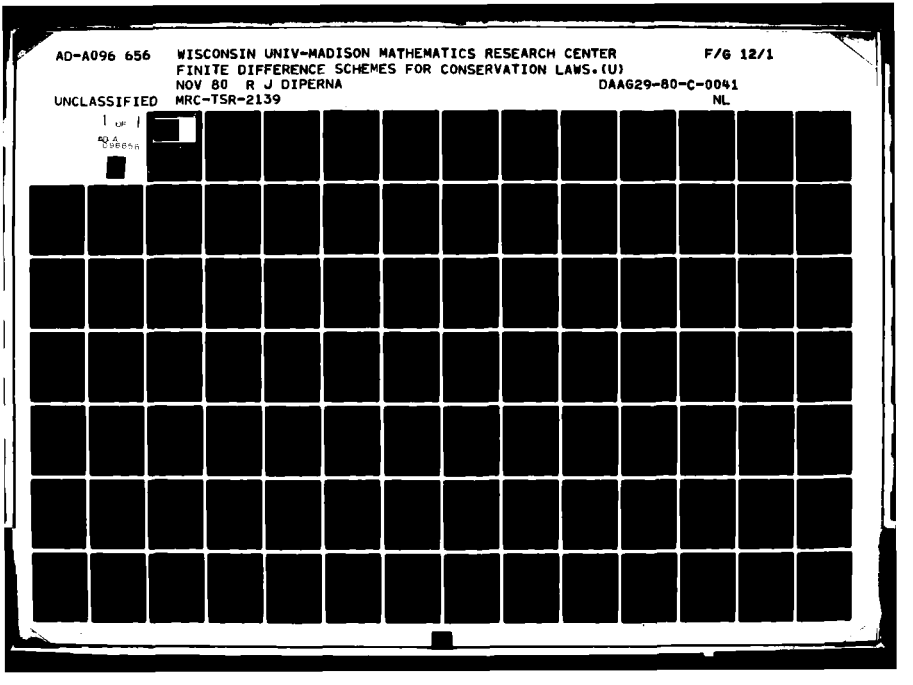

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FINITE DIFFERENCE SCHEMES FOR CONSERVATION LAWS

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ABSTRACT

We study finite difference approximations to weak solutions of the Cauchy problem for hyperbolic systems of conservation laws in one space dimension. We establish stability in the total variation norm and convergence for a class of hybridized schemes which employ the random choice scheme together with perturbations of classical conservative schemes. We also establish partial stability results for classical conservative schemes. Our approach is based on an analysis of finite difference operators on local and global wave configurations.

AMS (MOS) Subject Classifications: 35L65, 35L66

Key Words: finite difference schemes, conservation laws, Cauchy problems, shocks and singularities

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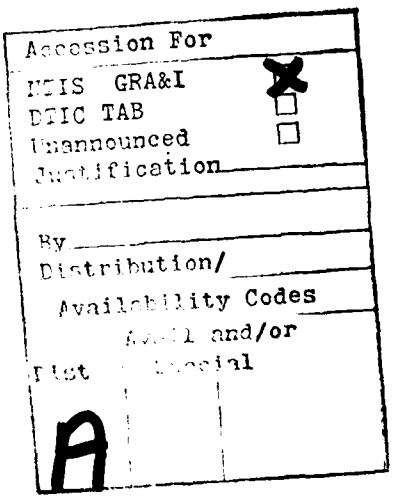
SIGNIFICANCE AND EXPLANATION

We are concerned with the numerical computation of shock waves using finite difference schemes. Specifically, we study problems concerning the stability and convergence of finite difference approximations and problems of describing the propagation of physical and numerical waves.



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FINITE DIFFERENCE SCHEMES FOR CONSERVATION LAWS

Ronald J. DiPerna*

1. INTRODUCTION.

We are concerned with finite difference approximations to weak solutions of the Cauchy problem for hyperbolic systems of conservation laws of the form

$$u_t + f(u)_x = 0, \quad -\infty < x < \infty . \tag{1.1}$$

Here the solution $u = u(x,t)$ takes on values in R^n and f is a smooth nonlinear mapping from R^n to R^n. We assume that the system is strictly hyperbolic in the sense that the Jacobian matrix $\nabla f(u)$ has n real and distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u) ,$$

and we require that each eigenvalue λ_j is either genuinely nonlinear or linearly degenerate in the sense of Lax [20], i.e. either

$$r_j \cdot \nabla \lambda_j \neq 0 \quad \text{or} \quad r_j \cdot \nabla \lambda_j \equiv 0 \tag{1.2}$$

for each index j where $r_j = r_j(u)$ denotes the corresponding right eigenvector of $\nabla f(u)$. Systems with this structure arise in several branches of continuum mechanics: fluid dynamics, MHD, elasticity, etc.

Experience with (1.1) has indicated that the space BV of functions of bounded variation provides a natural setting for the solution operator. It is known for example that if the initial data $u_0(x)$ lie in a small neighborhood of a fixed state $\bar{u} \in R^n$ and have small total variation, then a subsequence of the family of difference approximations $u(x,t,\Delta x)$ generated by the random choice method of Glimm converges pointwise a.e. to a solution u [12].

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Moreover, the entire family is stable in the total variation norm in the sense that

$$\text{(1.3)} \qquad TVu(\cdot,t,\Delta x) \leq \text{const } TVu_0(\cdot) ,$$

where the constant depends only on f and $\bar{u}$; corresponding estimates hold in the limit for u. In the case of data with large total variation, analogous stability and convergence results for the random choice method have been obtained for certain special systems [1, 7, 8, 17, 24, 25, 28, 29]. These results motivate the problem of determining the mechanisms which induce or preclude stability in the total variation norm for standard finite difference schemes, i.e. schemes which are conservative in the Lax-Wendroff sense [22].

We note that the problem of establishing stability for conservative difference schemes in any of the natural spaces for (1.1), e.g. BV, L^∞, L^2 and convergence remains open except in the case of first order accurate methods applied to scalar conservation laws [5, 30, 34]: in the case $n = 1$ the structure of the equation induces maximum principles for the corresponding exact solution operator in BV and L^p, $1 \leq p \leq \infty$; these maximum principles are preserved by the difference operators of those schemes which are precisely first order accurate. With regard to L^2-stability and its relationship to proper entropy production, we refer the reader to the work of Majda and Osher (27] on second order accurate schemes applied to scalar equations. In connection with the related role of L^∞-stability and action of the exact solution operator in the weak topology, we reference the work of Tartar [32] on the theory of compensated compactness, which contains several convergence results for exact solutions to general scalar conservation laws and their associated parabolic regularizations. In the computational setting, we refer the reader to the work of Chorin [2, 3, 4] on the implementation of the random choice method for reacting (and non-reacting) gas flow, to the work of Glimm, Marchesin and

McBryan [14, 15, 16] on hybridized approaches involving the random choice method together with procedures of subgrid refinement and wave tracking, to Crandall and Majda [35] on fractional step methods and to Engquist and Osher [36] on one-sided difference approximations.

In this paper we are primarily concerned with theoretical aspects of stability in the total variation norm and convergence for general finite difference schemes for systems of equations. We note that stability of the form (1.3) for a family of approximate solutions guarantees the existence of a subsequence converging pointwise a.e., since equations of the form (1.1) link the temporal and spatial variations of u; convergence of the entire family follows from uniqueness, in those particular circumstances where uniqueness is available [9]. One may, of course, entertain growth estimates on the total variation norm which are uniform in the mesh length.

We begin in Section 2 by formulating a new class K of difference schemes which are conservative in the Lax-Wendroff sense; the class K arises from a discrete approximation to the contour integral form of system (1.1) taken with respect to parallelograms having space-like sides in the x-t plane. The standard conservative schemes with a three-point domain of dependence can be subsumed by K after introducing a fractional step. In Section 8 we introduce a new class of hybridized schemes which employ the random choice method to approximate shock waves and perturbations of a certain class N of first order accurate schemes in K to approximate the continuous regions of flow; these hybridized schemes are based on the tracking of waves whose magnitudes lie between two specified thresholds depending on the mesh length. In the case of initial data with small total variation, we establish stability in the total variation norm and pointwise a.e. convergence of the difference approximations generated by the hybridized schemes applied to a class of systems of two

equations, cf. Sections 10 and 11. This class includes systems of the form

$$v_t + p(w)_x = 0, \quad w_t + q(v)_x = 0 \tag{1.4}$$

where $p'q' > 0$, e.g. the isentropic equations of gas dynamics and the equations for thin elastic beams in Lagrangian coordinates. In Section 12, we verify that the solutions constructed by these hybridized methods satisfy the entropy condition of Lax [21]. For general systems of n equations we obtain certain partial results concerning stability in the total variation norm for the aforementioned subclass N of first order accurate conservative schemes; the subclass N includes the Lax-Friedrichs scheme, cf. Section 4. For both the conservative and hybridized schemes, the total variation estimates are obtained with the aid of non-monotone functionals which are equivalent to the total variation norm, cf. Sections 4, 5 and 9.

The form of these functionals for schemes in class N is motivated by an analysis of the corresponding difference operators on discrete wave interactions. In Sections 3 and 4 we describe a general approach to the problem of analyzing difference operators on local and global wave configurations and apply it to the subclass N. For the purpose of analyzing the local action, we formulate a working notion of local discrete wave interaction which is based on a process of interpolating elementary waves between adjacent mesh points. We then study the relations which govern the magnitudes of the incoming and outgoing waves in a local interaction, cf. Sections 3, 5 and 7. In order to describe the global action, we introduce a state space Ω of global wave configurations X and associate, with a given scheme, a mapping

$$M : \Omega \to \Omega$$

such that each of the difference approximations u generated by the scheme corresponds to a discrete trajectory of the form

$$\{M^k X_0 : k = 0,1,2,\ldots\}$$

when X_0 represents the initial data, cf. Section 4. The generic state X in Ω consists of a sequence of local states x^k which record the structure of the local waves in u. Specific local states x^k are associated with a given finite difference approximation u by applying an interpolation map

$$I : R^n \times R^n \to R^n$$

to pairs of values of u at adjacent mesh points; the map I transforms a pair (v,w) in $R^n \times R^n$ into a set

$$\{\varepsilon_1(v,w),\ldots,\varepsilon_n(v,w)\}$$

of elementary wave magnitudes $\varepsilon_j(v,w)$ associated with the classical solution of the Riemann problem for (1.1) with Riemann data (v,w). For concreteness, we restrict our attention in this paper to three-level schemes which employ a stencil based on four mesh points; we note that the Lax-Friedrichs scheme and the random choice scheme can be regarded as having this form, while the Lax-Wendroff scheme can be viewed as a composition of two such schemes. For stencils with that geometry, the associated space Ω consists of states $X = x^k$ where x^k lies in R^{3n} and assumes the form

$$x^k = (\delta,\gamma,s) .$$

Here s represents the value of the difference approximation u at a specified mesh point (x,t) while $\delta = (\delta_1,\ldots,\delta_n)$ and $\gamma = (\gamma_1,\ldots,\gamma_n)$ denote sets of interpolated wave magnitudes associated with the mesh points immediately to the left and right of (x,t).

In the framework of the space Ω, we formulate working definitions of approximate simple wave and weakly interacting state and we identify several classes A of such states which are attracting from the marching map M of schemes in class N in the sense that the restriction of the orbit $M^k X$ to A^C runs down-hill with respect to associated "coercive" potential functionals Q_A. The reduction which these potentials Q_A experience in one time-step

acting on states X in A^c exceeds the corresponding increment to the total variation norm due to numerical and/or physical wave amplification. This property yields a uniform estimate on the total variation norm for those segments of the orbit which lie in A^c and have an initial point with small total variation. This partial stability result for schemes in class N serves as the starting point of our analysis of the hybridized schemes.

Preliminary to the construction of the potential functionals, we compare in Section 3 the behavior of the exact solution with that of the difference operators in class N on local wave interactions. In the setting of Ω, a discrete interaction consists of a collection of pairs of mesh points together with the corresponding sets of interpolated wave magnitudes. In the case of waves with small magnitude, a Taylor expansion of the equations relating the incoming and outgoing magnitudes produces a set of dominant terms with a fairly clear numerical interpretation; a comparison with the corresponding expansion derived by Glimm [12] for random choice interactions or, what is the same up to quadratic terms, exact wave interactions reveals several numerical mechanisms which are absent in the exact solution operator. As a preliminary step in the direction of classifying the numerical modes of wave propagation, we discuss in the setting of class N several numerical processes which we refer to as self-interaction, splitting and incomplete cancellation, cf. Section 3. These processes are reflected in the structure of the potential functionals Ω_A.

The motivation for a general study of potential functionals in the context of conservative difference schemes is the following. We recall that, for the exact solution operator, wave interactions typically increase wave magnitudes: an exact solution $u(x,t)$ to a system of equations generally admits a countable set of times t_n such that

$$\lim_{t \downarrow t_n} TVu(\cdot,t) > TVu(\cdot,t_n) .$$

On the other hand, in the setting of the random choice method, Glimm demonstrated that a potential for wave interaction can be attributed to each wave configuration through a quadratic functional $Q_{rc}(u)$ in such a way that all weak interactions reduce Q_{rc} more than they augment the total variation norm [12]. The potential Q_{rc} is quadratic in the sense that

$$c_1(TVu)^2 \leq Q_{rc}(u) \leq c_2(TVu)^2 ,$$

while non-increasing and compensating in the sense that

$$Q_{rc}\{u(\cdot,t,\Delta x)\} \quad \text{and} \quad F(u) \equiv TVu(\cdot,t,\Delta x) + Q_{rc}\{u(\cdot,t,\Delta x)\}$$

are both nonincreasing functions of time, if the initial data of the random choice approximations $u(x,t,\Delta x)$ have small total variation. The structure of Q_{rc} is discussed in Section 7. The stability estimate (1.3) follows from the equivalence of F and TV on small data.

These results motivate the problem of constructing potentials for standard difference schemes which compensate for both the physical and numerical amplification waves. Now, in the setting of conservative difference schemes two new features arise. The first is associated with the existence of numerical modes of wave propagation; it is not difficult to show for example that, as a consequence of augmented wave amplification, there exist no compensating potentials which are monotone and depend, as Q_{rc} does, only on the magnitudes of waves in a given configuration. The second is associated with the existence of a substantial class of states $\tilde{X}$ representing shock profiles which are reproduced by the scheme after a finite number of time steps module a spatial translation, i.e.

$$M^p\tilde{X} = \tilde{X}_q; \ \tilde{X}_q = \{\tilde{X}^{k+q}\} .$$

Clearly any translation invariant monotone function must be constant along the entire orbit corresponding to each shock profile $\tilde{X}$. We note that the existence of shock profiles for a broad class of conservative schemes has been established

by Majda and Ralston [28] in the context of systems of equations and by Jennings [18] in the context of scalar equations; numerical evidence has indicated that such states are stable.

The existence of numerical wave amplification and shock profiles motivates the study of functionals which appeal to the geometric strucuture of wave configurations in addition to information on their individual wave magnitudes and which are non-monotone when restricted to orbits $M^k X$. In particular it leads one to ask if there exist special classes A of states containing the shock profiles and corresponding potentials Q_A which are coercive on A^C in the sense that

$$Q_A(MX) - Q_A(X) \leq -\Delta_A^2(X) \tag{1.5}$$

if $X \in A^C$, where $\Delta_A(X)$ denotes the distance from X to A in some metric on Ω and compensating on A^C in the sense that

$$TV(MX) + Q_A(MX) \leq TV(X) + Q_A(X) \tag{1.6}$$

if $X \in A^C$. If such potentials exist and if the scheme under consideration is in fact convergent, one might expect that the structure of states in A and/or the coercive behavior of Q_A would permit only a mild growth independent of the mesh length for the functional $F = TV + Q_A$ along the entire orbit.

In Section 5 we construct potentials Q_A for the class N where the role of A is played by certain classes of approximate simple waves and by certain classes of weakly interacting states. The functionals Q_A are not monotone when restricted to the orbit $M^k X$ but do exhibit a rather strong coercive behavior on A^C, satisfying inequalities of the form (1.5) and (1.6) on A^C; here the quantity $\Delta_A^2(X)$ does not arise exactly as the square of a distance from X to A in a fixed metric on Ω but rather involves the square of a variable distance from X to a subclass of A. We conjecture that, along the entire orbit, the corresponding functionals $F = TV + Q_A$ for schemes in class

N experience only a mild growth independent of the mesh length. For the hybridized schemes we show that this is in fact the case by appealing to the improved resolution of local wave interactions which hybridization affords.

2. CONSERVATIVE DIFFERENCE SCHEMES.

In this section we formulate a new class of difference schemes which are conservative in the Lax-Wendroff sense [22]; the motivation is the following. Suppose $u = u(x,t)$ is a distributional solution in $BV \cap L^{\infty}$ to a system of conservation laws (1.1): the vector-field $(f(u),u)$ *is divergence-free in the* sense that the sum of the measures u_t and f_x vanishes on all Borel sets B,

$$\{u_t + f(u)_x\}(B) = 0 \ . \tag{2.1}$$

Green's theorem for measures [10, 33] yields an equivalent formulation by requiring that the integral of the normal component of (f,u) vanish for all piecewise smooth closed contours C:

$$\int_C \nu_t u + \nu_x f(u) ds = 0 \tag{2.2}$$

where $\nu = (\nu_t,\nu_x)$ denotes, for concreteness, the outward unit normal to C and ds the element of arc length. Indeed, (2.2) implies (2.1) provided only that C lies within a substantial class of contours, for example, parallelograms with sides parallel to two fixed directions.

Classical conservative schemes correspond to a discrete approximation of (2.1) with C taken as the boundary of a rectangle with sides parallel to the axes: for example, the standard conservative schemes with a three-point domain of dependence employ a grid with mesh points of the form $(i\Delta x, j\Delta t)$, i and j arbitrary integers, and generate the value of the difference approximation, say u, at a typical mesh point (x,t) in terms of the three known values immediately below

(2.3) $u(x,t) = \psi\{u(x - \Delta x, t - \Delta t),\ u(x, t - \Delta t),\ u(x + \Delta x, t - \Delta t)\}$;

the generating function ψ depends on the choice of mesh lengths Δx and Δt and is derived from a discrete approximation to (2.1) with C taken as the boundary of the rectangle

$$\{(x,t) : i\Delta x < x < (i + 1)\Delta x,\ j\Delta t < t < (j + 1)\Delta t\} .$$

In this section we shall describe a class of conservative schemes based on a grid having a diamond-shaped stencil, i.e. mesh points of the form $(i\Delta x, j\Delta t)$ where i and j are integers such that $i + j$ is even. We begin by describing a class of three-level schemes where generating function ϕ is derived from a discrete approximation to (2.1) with C taken as the boundary of a rhombus D with vertices at three time-levels of the form

(2.4) $n = (x,t),\ s = (x,t-2\Delta t),\ w = (x-\Delta x, t-\Delta t),\ e = (x+\Delta x, t-\Delta t)$,

where (x,t) is a typical mesh point: the value u_n of the difference approximation at the north vertex n is generated from known values at the west, south and east vertices by a formula of the form

$$u_n = \phi(u_w, u_s, u_e) ,$$

where ϕ depends on the ratio of mesh lengths $\Delta x/\Delta t$. To be precise, let $\alpha = (\alpha_t, \alpha_x)$ and $\beta = (\beta_t, \beta_x)$ denote respectively the outward unit normals to the ne side (northeast) and the wn side of the rhombus D with verticies (2.3). The normals α and β depend only on the fixed ratio of mesh lengths $\Delta x/\Delta t$. We introduce two smooth mappings

$$H : R^n \times R^n \to R^n \quad \text{and} \quad G : R^n \times R^n \to R^n$$

which respectively reduce along the diagonal $a = b$ to the normal component of the vector $(f(u),u)$ in the directions α and β:

(2.5)
$$H(a,b) = \alpha_t a + \alpha_x f(a) + p(a,b)(a - b)$$
$$G(a,b) = \beta_t a + \beta_x f(a) + q(a,b)(a - b) .$$

Here p and q simply denote arbitrary smooth $n \times n$ matrices; appropriate restrictions will be placed on p and q below. Next, we introduce the following formal approximations to the four line integrals whose sum represents the contour integral around the rhombus D:

$$H(u_n,u_e)\Delta s \sim \int_{ne} \alpha_t u + \alpha_x f(u)ds;\ G(u_n,u_w)\Delta s \sim \int_{wn} \beta_t u + \beta_x f(u)ds$$
$$-H(u_w,u_s)\Delta s \sim -\int_{ws} \alpha_t u + \alpha_x f(u)ds;\ -G(u_e,u_s)\Delta s \sim -\int_{se} \beta_t u + \beta_x f(u)ds\ ,$$

where Δs denotes the length of the sides of D. Summing the formulas above yields a formal approximation of the contour integral (2.2) with C replacing by D; the sum becomes an equation of the form

$$H(u_n,u_e) + G(u_n,u_w) - H(u_w,u_s) - G(u_e,u_s) = 0\ , \tag{2.6}$$

after the common coefficient Δs is factored out. Finally we assume that (2.6) can be solved for u_n in terms of the remaining variables yielding a smooth generating function ϕ:

$$u_n = \phi(u_w,u_s,u_e)\ .$$

For local purposes, solvability is guaranteed by requiring that the matrix

$$H_a(a,a) + G_a(a,a)$$

be invertible. For the centered schemes described above we have

$\alpha_t = \beta_t$, $\alpha_x = -\beta_x$ and

$$H_a(a,a) + G_a(a,a) = 2\alpha_t I + p(a,a) + q(a,a)$$

which is invertible if for example $p + q$ is small on the diagonal.

In a similar fashion, one can construct non-centered conservative schemes based on a discrete approximation to (2.2) with C taken as the boundary of a parallelogram with vertices of the form

$n = (x,t)$, $s = (x + \varepsilon\Delta x, t - 2\Delta t)$, $w = (x + \Delta x, t - \Delta t)$, $e = (x - \delta\Delta x, t - \Delta t)$.

More generally, one can construct a class K of multi-level conservative

schemes based on a discrete approximation to (2.2) with C taken as the boundary of a parallelogram intersecting several time-levels. However, we shall restrict our attention for concreteness to the subclass of centered three-level schemes based a stencil with vertices of the form (2.4). We note that this subclass contains several classical schemes. The leap-frog scheme is formed from an arithmetic average,

$$H(a,b) = \alpha_t(a + b)/2 + \alpha_x\{f(a) + f(b)\}/2$$

$$G(a,b) = \beta_t(a + b)/2 + \beta_x\{f(a) + f(b)\}/2 \ .$$

The Lax-Friedrichs scheme is obtained by eliminating the dependence on u_s, i.e. by taking $p = q = 0$. The general three-point conservative scheme (2.3) can be regarded as a composition of two schemes in this subclass by introducing a fractional step to produce a nine-point stencil having mesh points on five levels of the form

$$(x,t),(x,t \pm \Delta t),(x \pm \Delta x,t),(x \pm \tfrac{1}{2}\Delta x,t + \tfrac{1}{2}\Delta t),(x \pm \tfrac{1}{2}\Delta x,t - \tfrac{1}{2}\Delta t) \ .$$

Indeed, the standard two-step Lax-Wendroff scheme is already in this form since it can be regarded as a composition of the Lax-Friedrich and leap-frog schemes.

3. DISCRETE WAVES.

In this section we shall describe a method for introducing local wave magnitudes into a finite difference approximation. We shall present the method in the setting of the class K_1 of three-level schemes with a centered diamond shaped stencil; it has an obvious analogue for more general stencils. We shall also discuss the equations which relate the incoming and outgoing magnitudes of a local interaction and compare them with the corresponding equations for the exact solution and the random choice method.

A pattern of local wave magnitudes can be associated with a difference approximation defined on a grid by using the classical solution of the Riemann

problem [20], i.e. the initial value problem with data of the form

$$u_0(x) = u^- \text{ if } x < 0,\ u_0(x) = u^+ \text{ if } x > 0 ,$$

where u^- and u^+ denote elements of R^n. We recall that the exact solution operator resolves Riemann data (u^-,u^+) into a similarity solution $u = u(x/t)$ consisting of $n + 1$ constant states u_j, $j = 1,2,\ldots,n + 1$, with adjacent constant states separated by either a j-shock wave or a centered j-rarefaction wave [20]. Here $u_1 = u^-$ and $u_{n+1} = u^+$. In the standard fashion, we take the magnitude of a j-shock wave separating states u_j and u_{j+1} to be the negative of the distance from u_j to u_{j+1} along the j-shock wave curve through u_j and the magnitude of a centered j-rarefaction wave separating states u_j and u_{j+1} to the (positive) distance from u_j to u_{j+1} along the j-rarefaction wave curve through u_j. For example, the solution u to the Riemann problem for a system of two equations might consist of a 1-shock $x = \sigma t$ separating states u^- and u_2 together with a centered 2-rarefaction wave separating u_2 and u^+, i.e. $u = u^-$ in $-\infty < x/t < \sigma$, $u = u_2$ in $\sigma < x/t < \lambda_2(u_2)$, in $u = u^+$ in $\lambda_2(u^+) < x/t < \infty$ and the section $\lambda_2(u_2) < x/t < \lambda_2(u^+)$ forms a centered 2-rarefaction wave. In general we shall denote by

$$\varepsilon_j = \varepsilon_j(u^-,u^+)$$

the magnitude of the j-wave in the solution of the Riemann problem with data (u^-,u^+).

We shall restrict our attention to the class K_1 of three-level schemes in K based on a grid having a centered diamond-shaped stencil with vertices of the form (2.4). Consider a corresponding difference approximation $u = u(x,t,\Delta x,\Delta t)$ with small oscillation and suppose that the Courant-Friedrichs-Levy condition is satisfied, i.e.

$$\Delta x/\Delta t > \max\{|\lambda_j(v)| : j = 1,2,\ldots,n\} ,$$

where the maximum is taken over a set in R^n containing the range of the

difference approximation u. We shall associate with u a pattern of wave magnitudes by interpolating the solution of the Riemann problem between adjacent mesh points in the following fashion: associate with each pair

$$(p_1,p_2) = \{(x,t),(x \pm \Delta x, t \pm \Delta t)\}$$

of adjacent mesh points, the set of magnitudes of the n elementary waves in the solution to the Riemann problem with data

$$u^- = u(p_1),\ u^+ = u(p_2)\ .$$

This association can be expressed formally by the map

$$I : R^n \times R^n \to R^n$$

$$I(u^-,u^+) = \{\varepsilon_1(u^-,u^+),\ldots,\varepsilon_n(u^-,u^+)\}\ .$$

By a <u>local wave interaction</u> in u we shall mean a configuration consisting of a mesh diamond having verticies n,s,e and w of the form (2.4) together with the four sets of wave magnitudes which are obtained from the Riemann problems associated with the four pairs of adjacent verticies $(w,s),(s,e),(w,n)$ and (n,e) and which are denoted as follows:

$$\delta = (\delta_1,\ldots,\delta_n) = I(u_w,u_s) \quad \gamma = (\gamma_1,\ldots,\gamma_n) = I(u_s,u_e)$$

$$\alpha = (\alpha_1,\ldots,\alpha_n) = I(u_w,u_n) \quad \beta = (\beta_1,\ldots,\beta_n) = I(u_n,u_e)\ .$$

We shall refer to δ_j and γ_j as the <u>incoming j-waves</u> and to α_j and β_j as the <u>outgoing j-waves</u>.

We note that one may also interpolate between adjacent mesh points even if the C-F-L condition is not satisfied. In this case, however, it is not so clear how to interpret the interpolation. On the other hand, if the C-F-L condition holds, it is meaningful to interpolate between mesh points on the same space-like arc as well as between points on the same time-level, since the exact solution operator applied to Riemann data is invariant under a space-like rotation of the x-t plane. Indeed, the C-F-L condition guarantees that the line segments ws,se,wn and ne forming the boundary of a typical mesh diamond

are space-like for difference approximatins with small oscillation. We remark that one is free to imagine a set of n elementary waves actually crossing the line segment joining a pair of adjacent mesh points (p_1,p_2); waves having a location specified only within a distance $|p_1-p_2|$. However, we emphasize that by definition we only associate a set of n wave magnitudes with a given pair (p_1,p_2). With this understanding, one can summarize a discrete wave interaction, say for a system of two equations such as Figure 1 of Section 4 where the symbols u_k,s_k,u_{k+1} and w_k denote the values of the difference approximation at the corresponding diamond indexed by k.

We shall begin the discussion of local interactions by considering the relationship between the incoming and outgoing waves. It is clear that the outgoing magnitudes α and β are smooth functions of the incoming magnitudes δ and γ and the local base state u_s:

$$(\alpha,\beta) = W(\delta,\gamma,u_s)$$

We note that δ,γ and u_s uniquely determine u_w and u_e which together with u_s uniquely determine u_n,α and β through the smooth generating function ϕ. A Taylor expansion of W at $\delta = \gamma = 0$ provides the dominant terms in the laws governing the interaction of weak waves. To begin with we let

$$\nu_j = (\delta_j,\gamma_j) \quad \text{and} \quad \sigma_j = (\alpha_j,\beta_j)$$

denote the incoming and outgoing j-waves and write

$$\sigma = A(u_s)\nu + O(|\nu|^2) \tag{3.1}$$

where $A(u_s)$ is a smooth $2n \times 2n$ matrix and $\nu = (\nu_1,\ldots,\nu_n)$, $\sigma = (\sigma_1,\ldots,\sigma_n)$.

For simplicity we shall restrict our attention to a broad subclass of schemes in K_1 which preserves two basic properties of the exact solution operator. In this connection, we first recall that if all incoming waves of an interaction in an exact solution belong to the same characteristic field, say

the jth field, then the magnitudes of the outgoing waves of the kth field, $k \neq j$ are quadratic with respect to the magnitudes of the incoming waves. We shall, first of all, restrict attention to those schemes in K_1 which preserve this property, i.e. schemes such that

$$\sigma_k = O(|\nu_j|^2), \quad \text{if} \quad \nu_k = 0 \quad \text{for} \quad k \neq j \ . \tag{3.2}$$

Clearly this property is equivalent to the statement that $A(u_s)$ is a tridiagonal matrix for all u_s. We note that property (3.2) is satisfied by the Lax-Friedrichs scheme and the leap-frog scheme; a simple criterion for (3.2) is given in a lemma below in the setting of schemes in K_1 and can easily be checked for the Lax-Firedrichs and leap-frog schemes. It follows as a corollary that the two-step Lax-Wendroff scheme satisfies the natural analogue of (3.2) since it can be viewed as the composition of the Lax-Friedrichs scheme and the leap-frog scheme; indeed, the same is true for all the standard variations on the Lax-Wendroff scheme since they share the same linearization. Finally, we remark that the random choice method can be regarded as a scheme which employs the same stencil as schemes in K_1 and it satisfies (3.2); the laws for random choice interactions are recalled in Section 7.

Secondly, we recall that wave interactions in an exact solutin do not augment wave magnitudes by more than a quantity which is quadratic with respect to the magnitudes of the incoming waves. We shall restrict our attention further to those schemes in K_1 which preserve this property, i.e. schemes such that

$$\sum_{j=1}^{n} |\alpha_j| + |\beta_j| \leq \sum_{j=1}^{n} |\delta_j| + |\gamma_j| + O(|\nu|^2) \ . \tag{3.3}$$

The condition (3.3) is equivalent to the condition that

$$\begin{aligned} \alpha_j &= (1 - \mu_j)\delta_j + \tau_j\gamma_j + O(|\nu|^2) \\ \beta_j &= \mu_j\delta_j + (1 - \tau_j)\gamma_j + O(|\nu|^2) \ , \end{aligned} \tag{3.4}$$

where the coefficients μ_j and τ_j satisfy

(3.5) $$0 \leq \mu_j(u_s) \leq 1, \quad 0 \leq \tau_j(u_s) \leq 1 ,$$

for all values of their argument u_s. Now, it is not difficult to show that, within the subclass of schemes in K_1 which satisfy property (3.2), inequality (3.3) holds for those methods which are precisely first order accurate. In this connection we recall that any scheme which is consistent with the equations and which has a smooth generating function is at least first order accurate. Thus, condition (3.3) rules out second order accurate methods. In particular, the Lax-Friedrich scheme satisfies (3.3) while the leap-frog and Lax-Wendroff schemes do not. It is also simply to verify that the random choice method satisfies (3.3), cf. Section 7. Finally, we remark that for certain technical reasons we shall restrict our attention to the subclass N of schemes in K_1 which satisfy (3.2), (3.3) and

(3.5) $$0 < \mu_j(\tau_s) < 1, \quad 0 < \tau_j(u_s) < 1 .$$

The lemma below contains a simple criterion for membership in N which shows in particular, that the Lax-Friedrichs scheme belongs to N.

Lemma. Consider a scheme in class K. The corresponding matrix A is tridiagonal if and only if the matrix

$$w(a) = p(a,a) + q(a,a) ,$$

obtained from (2.3), satisfies

(3.6) $$w(a)r_j(a) = w_j(a)r_j(a), \quad j = 1,2,\ldots,n ,$$

where r_j denotes the right eigenvector of f' associated with the eigenvalue λ_j. If (3.5) holds then

(3.7) $$\mu_j = \{\alpha_t + \alpha_x\lambda_j(a) + w_j(a)\}/\{\sigma_t + \sigma_x\lambda_j(a) + w_j(a)\}$$

(3.8) $$\tau_j = \{\beta_t + \beta_x\lambda_j(a) + w_j(a)\}/\{\sigma_t + \sigma_x\lambda_j(a) + w_j(a)\}$$

where $\sigma_t = \alpha_t + \beta_t$, $\sigma_x = \alpha_x + \beta_x$ and the scheme lies in N if and only if

the eigenvalues $w_j(a)$ are such that the quantities specified by (3.7) and (3.8) lie strictly between zero and one.

Proof: Consider an iteraction associated with a mesh diamond such that $u_s = u_e$ and u_w lies on the j-wave curve through u_s. In this case the only incoming wave is a j-wave crossing the ws side of the diamond. Regard u_s as fixed and u_w as parametrized by arc length δ along the j-wave curve through u_s:

$$u_w = u_w(\delta), \quad u_w(0) = u_s .$$

Substituting $u_w = u_w(\delta)$ and $u_s = u_e$ into (2.5), solving for $u_n = u_n(\delta)$ and differentiating with respect to δ at $\delta = 0$ yields

$$\dot{u}_n(0) = (H_a + G_a)^{-1}(H_a - G_a)\dot{u}_w(0) , \tag{3.9}$$

where the coefficient matrix is evaluated at (u_s,u_s). A simple calculation shows that the matrix in (3.9) is given by

$$\{\sigma_t I + \sigma_x f'(a) + w(a)\}^{-1}\{\alpha_t I + \alpha_x f'(a) + w(a)\}$$

where $a = u_s$. By considering the analogous incoming configuration where $u_w = u_s$ and u_e lies on the j-wave curve through u_s we obtain a corresponding equation

$$\dot{u}_n(0) = (H_a + G_a)^{-1}(G_a - H_b)\dot{u}_e(0) ,$$

in which the coefficient matrix is given by

$$\{\sigma_t I + \sigma_x f'(a) + w(a)\}^{-1}\{\beta_t I + \beta_x f'(a) + w(a)\} ,$$

with $a = u_s$. The lemma follows from the fact that

$$\dot{u}_w(0) = \dot{u}_e(0) = r_j(u_s) .$$

Remarks: It follows from the lemma that a scheme in class N necessarily satisfies the C-F-L condition. Conversely, if a scheme in K has a tridiagonal matrix A then the C-F-L implies that μ_j and τ_j lie between zero and one provided $w_j(a)$ is sufficiently small. The latter fact applies to the Lax-Friedrichs scheme for which $p = q = 0$.

Next we shall describe several numerical mechanisms of interaction present in class N schemes. For this purpose, let us write the expansion (3.2) for a class N scheme in the form

$$\sigma_j = A_j(u_s)\nu_j + B_j(u_s)(\nu,\nu) + O(|\nu|^3)$$

$$B_j(u_s)(\nu,\nu) = \sum_{k\neq j} b^j_{jk}(u_s)(\nu_j,\nu_k) + \sum_{k=1}^{n} b^j_{kk}(u_s)(\nu_j,\nu_j)$$

where $\nu_j = (\delta_j,\gamma_j)$, $\sigma_j = (\alpha_j,\beta_j)$ and $b^j_{k\ell}$ are bilinear maps from $R^2 \times R^2$ to R^2 depending on the local base state u_s. The presence of numerical mechanisms of wave interaction is revealed by the structure of A_j and B_j.

Self-interactions. The structure of the operator B_j shows that the nonlinear interaction between characteristic fields in a difference approximation is substantially larger than in an exact solution. By way of example, let us consider an exact solution $\tilde{u}$ to a system of two equations which consists of two interacting weak shocks of different fields. To be precise, suppose that in a strip of the form $(0,t_0)$, $\tilde{u}$ consists of a 2-shock δ_2 and a 1-shock γ_1 which have trajectories

$$x - x_0 = s_\delta(t - t_0) \quad \text{and} \quad x - x_0 = s_\gamma(t - t_0),\ t < t_0$$

and which approach with speeds $s_\delta > s_\gamma$; while in a strip of the form (t_0,∞), $\tilde{u}$ consists of a 1-shock α_1 and a 2-shock β_2 which have trajectories

$$x - x_0 = s_\alpha(t - t_0) \quad \text{and} \quad x - x_0 = s_\beta(t - t_0),\ t < t_0$$

and which recede with speeds $s_\alpha < s_\beta$; u is constant in each of the four sectors defined by the four rays above. It is well-known the outgoing magnitudes satisfy

$$\alpha_1 = \gamma_1 + O(\gamma_1\delta_2);\ \beta_2 = \delta_2 + O(\gamma_1\delta_2)\ .$$

For systems of n equations, two interacting weak shocks δ_j and γ_k generate

two shocks α_j and β_k satisfying

$$\alpha_j = \gamma_j + O(\gamma_j\delta_k);\ \beta_k = \delta_k + O(\gamma_j\delta_k)$$

together with reflected waves

$$\varepsilon_\ell = O(\gamma_j\delta_k)$$

in the ℓ^{th} field, $\ell \neq j$, $\ell \neq k$. Thus, in these two examples the wave magnitudes are conserved in each characteristic field up to first order while the amplification of individual waves and production of reflected waves is at most on the order the product of the approaching waves. The same statement can be made for multiple wave interactions. Perhaps the simplest formulation is provided by the laws for multiple wave interactions in the randon choice cf. Section 7, these laws coincide with those of the exact solution up to and including quadratic terms. For the purposes of the present discussion we only want to remark that for the exact solution operator and random choice operator the diagonal terms of B_j vanish identically. In constrast, the diagonal terms of B_j for schemes in class N, i.e. the matricies $b^{(j)}_{kk}(u_s)$ do not vanish on any open set. The term

$$b^j_{kk}(u_s)(\nu_k,\nu_k) \tag{3.9}$$

records a contribution to the j^{th} field from the <u>self-interaction</u> of waves in the k^{th} field. If $k \neq j$ then the term (3.9) represents a contribution to the production of a reflected waves in the j^{th} field due to self-interactions in the k^{th} field. If $k = j$ then the term (3.9) represents a contribution to the amplification of waves in the j^{th} field due to self-interactions in the j^{th} field. For example, suppose that the incoming waves of a discrete interaction belong to the k^{th} field, i.e. $\nu_\ell = 0$, $\ell \neq k$. First, we see that outgoing waves σ_j are produced on the j^{th} field $j \neq k$ satisfying

$$\sigma_j = b^{(j)}_{kk}(u_s)(\nu_k,\nu_k) + O(|\nu_k|^3),\ j \neq k\ .$$

Thus, although there exist no incoming waves of the j^{th} field, $j \neq k$, there do exist outgoing "reflected" waves of the j^{th} field arising from self-interactions in the k^{th}-field. Secondly, the waves of the k^{th} field are themselves amplified by a term of the form (3.9), i.e.

$$\alpha_k + \beta_k = \delta_k + \gamma_k + b_{kk}^{(j)}(u_s)(\nu_k, \nu_k)^2 + O(|\nu_k|^3) .$$

Wave Splitting. Consider a discrete interaction with only one incoming wave, say δ_j. Up to linear terms, the action of a scheme in class N is to split δ_j into two waves in a proportion determined by the local base state u_s:

$$\alpha_j = \{1 - \mu_j(u_s)\}\delta_j + O(\delta_j^2); \ \beta_j = \mu_j(u_s)\delta_j + O(\delta_j^2) .$$

For a general interaction, each of the incoming waves is split and then superimposed up to linear terms in a fashion determined by the structure of the matrices A_j. The process of wave splitting is absent in the exact solution operator to systems with eigenvalues of the form (1.2). We remark, in passing, that if an eigenvalue λ_j is not monotone in the direction r_j then shocks in the exact solution can be split spontaneously through interactions with smooth flow. The process of wave splitting is also absent in the random choice method except for the trivial situation where a rarefaction wave is split by a sample point. In the random choice method the splitting of rarefaction waves is not accompanied by any form of wave amplification.

One of the interesting consequences of wave splitting a conservative scheme is that the recession of waves after interaction is not sharp. In the special exact solution $\tilde{u}$ described in the subsection on self-interactions, the two receding shock waves α_1 and β_2 from the boundary of an identically constant wake region. For a conservative scheme two "receding" shock waves are split again and again at each time level, leaving a wake regions with waves on the same order as the primary waves themselves.

Incomplete Wave Cancellation. The well-known persistence of oscillations observed in difference approximations generated by conservative difference schemes corresponds to an incomplete cancellation experienced by interpolated waves of the same characteristic field but of different sign, i.e. j-waves with positive and negative magnitude. In the random choice method, the interaction of a j-rarefaction wave (positive magnitude) and a j-shock wave (negative magnitude) leads to the absorption of the smaller wave by the larger up to linear terms. A similar statement can be made for exact solutions by considering the effect of such an interaction after a small interval of time. In contrast, in a conservative difference scheme the larger wave only absorbs a fraction of the smaller in typical interactions. An analytical discussion of this feature is postponed until Section 6.

4. *GLOBAL WAVE CONFIGURATIONS.*

In this section we shall describe a framework for studying local and global interactions in a finite difference approximation generated by a member of the class K_1 of three-level schemes with a centered diamond-shaped stencil. An analogous treatment suggests itself for schemes with more general stencils. To begin with, let us consider an arbitrary function u which is defined on a grid having a centered diamond-shaped stencil with mesh lengths Δx and Δt. Fix two consecutive levels $t = m\Delta t$ and $t = (m + 1)\Delta t$ and let (s_k) and (u_k) denote the sequences of values of u at the lower and upper levels; we put

$s_k = u\{(2k-1)\Delta x, m\Delta t\}$ if m is even and $s_k = u\{2k\Delta x, m\Delta t\}$ if m is odd,

$u_k = u\{2k\Delta x, (m+1\Delta t\}$ if m is even and $u_k = \{(2k+1)\Delta x, (m+1)\Delta t\}$ if m is odd,

and we put

$$\delta^k = I(u_k, s_k) \quad \text{and} \quad \gamma^k = I(s_k, u_{k+1}) .$$

Figure 1 illustrates the case $n = 2$. We note that the values s_k together with the interpolated magnitudes (δ^k, γ^k) uniquely determine the values u_k on the upper level. Thus, complete information for two consecutive levels is carried by the sequence

$$X = \{x^k\},\ x^k \equiv (\delta^k, \gamma^k, s_k) \in R^{3n} .$$

In certain circumstances it proves useful to regard each local state x^k as being decomposed into <u>local states of the j^{th} field x_j^k:</u>

$$x_j^k = (\delta_j^k, \gamma_j^k, s_k) .$$

Now in the case where u arises as a finite difference approximation associated with a scheme in K_1 having a generating function ϕ, the process of advancing u from one time-level to the next can be conveniently represented by the following map M defined on the set Ω of all such states X:

$$M\{(\delta^k, \gamma^k, s_k)\} = (\beta^{k-1}, \alpha^k, u_k)$$

where the outgoing wave magnitudes α^k and β^k are obtained from ϕ and the corresponding incoming waves by the rule

$$\alpha^k = I(u_k, w_k);\ \beta^k = I(w_k, u_{k+1})$$

where

$$w_k = \phi(u_k, s_k, u_{k+1}) ,$$

cf. Figure 1. Thus, the marching map M represents the process by which incoming magnitudes δ^k, γ^k and base states s_k determine values u_k on the next level, which in turn produce values w_k and outgoing magnitude through the generating function ϕ. With the aid of such sequences, we shall identify a given finite difference approximation u with the discrete trajectory of its data X_0 under M in Ω:

$$u \sim \{M^p X_0\}_{p=0}^{p=0} .$$

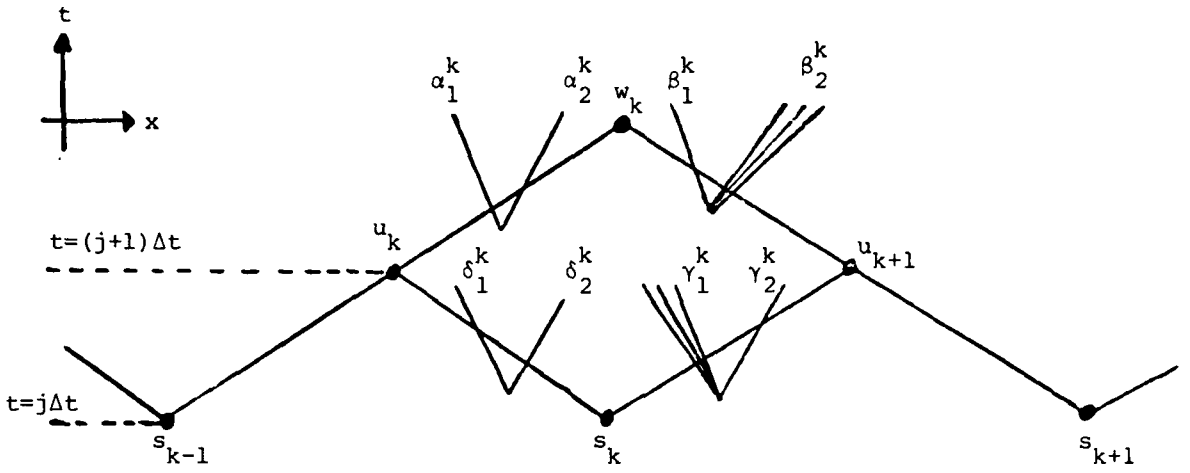

Figure 1

The problem is to prove that if TVX_0 is sufficiently small then for each time $T > 0$,

$$TVM^p x_0 \leq \text{const} .$$

if $p \leq T/\Delta t$, where the constant is independent of the mesh length. Here we define

$$TVX = \sum_{j,k} |\delta_j^k| + |\gamma_j^k| .$$

As we remarked in the introduction, the strategy is to study potentials for wave interaction. For the subclass N of schemes satisfying (3.2), (3.3) and (3.5), we shall construct appropriate potentials in several steps. The first is to introduce a notion of approximate simple wave as follows. Embed a diamond-shaped stencil, centered or not, with mesh lengths Δx and Δt into a j-simple wave $u = u(x,t)$ and examine the relationship between the wave magnitudes δ_j and γ_j obtained by applying the interpolation map I to the values of u at the west-south vertices and the south-east vertices respectively:

$$\delta = (\delta_1,\ldots,\delta_n) = I(u_w,u_s);\ \gamma = (\gamma_1,\ldots,\gamma_n) = I(u_s,u_e) .$$

A simple calculation reveals a restriction on the ratio of the magnitudes of the incoming j-waves:

$$\delta_j = \theta_j(u_s)\gamma_j + O(\Delta x^2 + \Delta t^2) ,$$

$$\delta_k = \gamma_k = 0, \; k \neq j ,$$

where θ_j denotes the ratio of direction cosines between the normals α and β associated with the stencil and the characteristic ray with speed $\lambda_j(u_s)$:

$$\theta_j(u_s) = \{\alpha_t + \alpha_x\lambda_j(u_s)\}/\{\beta_t + \beta_x\lambda_j(u_s)\} .$$

For the purpose of constructing potential functionals, local states $X_j = (\delta_j, \gamma_j, s)$ satisfying

$$\delta_j = \theta_j(s)\gamma_j$$

play a central role. We shall refer to

$$\Gamma_j(s) = \{X_j : \delta_j = \theta_j(s)\gamma_j\}$$

as the set of <u>local j-simple waves</u> passing through the base state $s \in R^n$ and we shall study the action of class N schemes on the corresponding global configurations

$$\Gamma(s) = \{X : \delta_j^k = \theta_j(s_k)\gamma_j^k\} .$$

One can regard an element of $\Gamma(s)$ as consisting of a chain of local simple waves (of varing index) passing through a sequence $s = (s_k)$ of local base states $s_k \in R^n$.

For schemes in class N we shall first construct a functional $P : \Omega \to R$ which decreases along those segments of the orbit M^pX which lie in the complement of a neighborhood of the simple waves in the sense that

$$P(MX) - P(X) \leq -cd_2(X) + ce_{\frac{1}{3}}(X) \tag{4.2}$$

if $TVX << 1$. Here and below we shall use the letter c to denote any of various positive constants which depend only on the system (1.1), the scheme

under consideration and the vector $\bar{u} \in R^n$ in the neighborhood of which all analysis takes place. The quantity $d(X)$ denotes the ℓ^2-distance from $X = \{(\delta^k, \gamma^k, s_k)\}$ to the corresponding set of simple waves $\Gamma(s)$, $s = (s_k)$, i.e.

$$d^2(X) = \sum_{j,k} \{\delta_j^k - \theta_j(s_k)\gamma_k^k\}^2/\{1 + \theta_j(s_k)\} ,$$

and $e_p(X)$ denotes the sum of the p^{th} powers of the wave magnitudes in X, i.e.

$$e_p(X) = \sum_{j,k} |\nu_j^k|^p .$$

One could, of course, normalize one of the constants c in (4.2) to equal one. In addition P is quadratic in the sense that

$$-c(TVX)^2 \leq P(X) \leq c(TVX)^2 .$$

These estimates motivate a study of a general class of neighborhoods of simple waves of the form

$$\Gamma_{mp} = \{X : d^2(X) \leq me_p(X)\} ,$$

which can be regarded as consisting of approximate simple waves. However, for the purposes of this paper the sets Γ_{mp} are needed only in the case where m is small and $p = 2$ and the case where m is large and $p = 3$. The case $p = 3$ is of particular interest since $e_3(X)$ represents the approximate rate of entropy production associated with the state X.

The details of the construction of P are postponed until Section 5; we shall presently restrict our remarks to certain qualitative properties of P. We begin with a simple observation that for schemes in class N the inequality (4.1) implies an estimate on the total variation along those segments of the orbit in Γ_{m2}^c. Indeed, for a scheme in class N, the interaction of weak waves augments the total variation norm at most quadratically, i.e.

$$TVMX \leq TVX + ce_2(X) ,$$

if as osc X << 1, where the oscillation of X is defined as the supremum of the absolute values of wave magnitudes δ_j^k and γ_j^k in X, and we obtain the following lemma.

Lemma. Given a scheme in class N and a constant $m > 0$, there exists a constant $c(m)$ such that the functinal $F_1 = TV + c(m)P$ satisfies

$$F_1(MX) \leq F_1(X)$$

if $X \in \Gamma_{m2}^c$ and if $TVX << 1$.

Since the functinals F_1 and TV are equivalent on states with small total variation, i.e.

$$cTVX \leq F_1(X) \leq cTVX$$

if $TVX << 1$, it follows as a corollary that

$$TVM^kX \leq cTVM^pX \quad \text{for} \quad p \leq k \leq q$$

if $TVM^pX << 1$ and if M^kX lie in Γ_{m2}^c for $p \leq k \leq q$. Granting the lemma, the problem of establishing stability in the total variation becomes one estimating the total variation norm along those segments of the orbit which lie near simple waves, i.e. in Γ_{m2}^c.

The Structure of P. The potential P is sum of n functionals P_j, each measuring the potential for interaction in waves of a given characteristic field. The functional P_j contains a constant coefficient quadratic potential for interaction as in the random choice potential [12] plus a weighted sum corresponding to numerical self-interactions:

$$P_j(X) = \sum_{D_j} \alpha\beta + \sum \phi_j(\alpha,u_\alpha)\alpha^2 . \tag{4.3}$$

Here $D_j = D_j(X)$ denotes the set of all pairs (α,β) of distinct j-waves in X. The weight $\phi_j(\alpha,u_\alpha)$ depends on the local base state u_α through which the

wave α passes, on orientation of the normal of the line segment which α crosses and on the way in which the scheme under consideration approximates a j-simple wave.

We note that, in contrast, the random choice potential contains terms of the form

$$\sum_{A_j} |\alpha\beta|$$

where A_j denotes the set of all pairs (α,β) of approaching j-waves in a given wave configuration; in the terminology of [12] a pair of j-waves is called approaching if at least member is a shock. The presence of terms of the form (4.4) is motivated by the fact that, in the random choice method, approaching j-waves α and β will collide in a finite time in the absence of interference from other waves, just as in an exact solution: if α and β are both shocks then the total variation typically increases at the point of interaction by a quantity on the order $|\alpha\beta|$; on the other hand if only one wave is a shock, the total variation is reduced by

$$-C(\alpha,\beta) + O(\alpha\beta) ,$$

where the cancellation between α and β is defined by

$$C(\alpha,\beta) = 2 \min(|\alpha|,|\beta|) \quad \text{if} \quad \operatorname{sgn}\alpha \neq \operatorname{sgn}\beta$$

$$= 0, \quad \text{otherwise} .$$

cf [12, 13]. In constrast, pairs of j-rarefaction waves in conservative difference approximations can also interact through numerical errors and one may expect their products to appear in potentials for wave interaction. Lastly, we remark that for class N schemes, one has the option of working with potential functionals P_j in which the leading term of (4.3) is replaced by

$$\sum_{D_j} |\alpha\beta| ,$$

but in this case additional terms involving cancellation effects $C(\alpha,\beta)$ appear on the right hand side of (4.2). Such potentials are useful in the context of hybridized scheme; for current purpose it suffices to discuss potentials of the form (4.3)

We shall now turn to the problem of estimating the total variation norm along those segments of the orbit which lie near simple waves. To this end one is led to study the local recession of waves after interaction; the motivation is the following. We recall that in the exact solution operator three main mechanisms of stability are present in the form of the cancellation process between shocks and rarefaction waves of the same field, in the spreading of individual rarefaction waves and in the recession of waves of different fields after interaction. Now, if a local state $X_j = (\delta_j, \gamma_1, s)$ lies close to a j-simple wave, for example, in the sense that

$$d^2(X_j) \leq m e_p(X_1)$$

with $p = 2$ and m small or with $p > 2$ and osc X_j small then clearly sgn δ_j = sgn γ_j and cancellation is absent. Secondly, one expects that the spreading of rarefaction waves will only be detected in the framework of a n-parameter interpolation after several time steps. Therefore, in studying the behavior of a class N scheme in two consecutive time steps acting on configurations near simple waves, it is natural to investigate the wave recession process. For this purpose we shall construct a functional T which measures the potential for transverse wave interactions and satisfies

$$T(MX) - T(X) \leq -c\tau(X) + cd^2(X) + ce_3(X) , \tag{4.5}$$

if $TVX \ll 1$. Here τ records the sum of products of all incoming transverse waves in X:

$$\tau(X) = \sum \{|\nu_j^k||\nu_\ell^k| : j < \ell \text{ and } -\infty < k < \infty\} .$$

In particular, if X lies in Γ_{m3} (for appropriate m) then T is reduced in one time step by a quantity on the order of τ modulo the approximate rate of entropy production, i.e.

$$T(MX) - T(X) \leq -c\tau(X) + ce_3(X) ,$$

if $TVX << 1$ and $X \in \Gamma_{m3}$. Thus if $X \in \Gamma_{m3}$ the functional T experiences virtually the same reduction, modulo e_3, as the exact solution operator. In this regard we note that τ includes products of both approaching and receding waves where, in the standard terminology, a pair (δ_j,γ_ℓ) is called <u>approaching</u> if δ_j lies to the left of γ_ℓ and either $j > \ell$ or $j = \ell$ and at least one member is a shock and <u>receding</u> if δ_j lies to the left of γ_ℓ and $j < \ell$. Now if a local state is "close" to the simple waves then τ is in fact on the order of the sum of just the products of those incoming waves which are approaching. The latter quantity is precisely the amount by which the potential for the random choice method is reduced in a local interaction cf. [12]. We conclude that for schemes in class N, a potential of the form $Q = cP + cT$ satisfies

$$Q(MX) - Q(X) \leq -c\tau(X) - cd^2(X) + ce_3(X) \tag{4.6}$$

if $TVX << 1$.

The form of the right hand side of (4.6) leads one to study the action of class N schemes on the following sets of states W_{mp} contained in Γ_{mp},

$$W_{mp} = \{X : \tau(X) + d^2(X) \leq me_p(X)\} .$$

If $p > 2$ or if $p = 2$ and m is small, one can regard a point X in W_{mp} as a <u>weakly interacting state</u> in the sense that the total amount of transverse wave interaction in X plus its distance squared to the corresponding simple waves is relatively small. In the case $p = 2$ we obtain a bound on the total variation along those segments of the orbit M^kX which lie in W_{mp}^c.

Lemma. Given a scheme in class N and a constant $m > 0$ there exist constants c_1 and c_2 depending on m such that the functional $F = TV + c_1P + c_2T$ satisfies

(4.7) $$F(MX) \leq F(X)$$

if $TVX << 1$ and if $X \in W^c_{m2}$.

Again, it follows as an immediate corollary that

(4.8) $$TVM^kX \leq cTVM^pX \quad \text{for} \quad p \leq k \leq q$$

if $TVM^pX << 1$ and if M^kX lies in W^c_{m2} for $p \leq k \leq q$.

In Section 6 we shall describe a procedure for perturbing the generating function of class N schemes in such a way as to damp the numerical reflected waves which are produced by the interactions of configurations near simple waves. For the perturbed schemes estimates of the form (4.7) and (4.8) hold provided the X and M^kX respectively lie in the complement of the much smaller third order states W_{m3}. As we remarked above the sets W_{m3} are of particular interest since $e_3(X)$ represents the approximate rate of entropy production.

Structure of T. The functional T consists of two terms. The first represents the standard quadratic potential for approaching waves of different characteristic fields and the second represents a numerical potential for self-interactions within groups of waves associated with pairs of adjacent mesh points:

$$T(X) = \sum_A |\alpha\beta| + \sum \psi(\alpha,\beta,u_\alpha,u_\beta)|\alpha\beta| \ .$$

Here $A = A(X)$ denotes the set of all pairs (α,β) of approaching waves of different fields and the weight ψ has the following property: if α is a j-wave and β a k-wave, $k \neq j$, crossing the west-south side or the south-east side of a mesh diamond with base state u_s respectively then

$$\psi = p_{jk}(u_s) \quad \text{or} \quad \psi = q_{jk}(u_s)$$

for certain smooth functions p_{jk} and q_{jk}, which characterize the numerical self-interactions among the waves of the associated Riemann problems, otherwise $\psi = 0$. The details of the construction of T are given in the next section.

5. POTENTIAL FUNCTIONALS.

In this section we shall construct the functionals P_j and T discussed in Section 4. The construction of P_j is based on the following fact. Consider the linear mapping of R^2 defined by

(5.1) $$\sigma = A\nu; \quad A = \begin{pmatrix} 1 - \mu & \tau \\ \mu & 1 - \tau \end{pmatrix}$$

where $\nu = (\delta,\gamma)$ and $\sigma = (\alpha,\beta)$. Such mappings arise from the leading term of (3.1) with a fixed value of the local base state u_s. A simple computation shows that quadratic form

$$Q(\nu) = a\gamma^2 + b\delta^2 + \gamma^\delta$$

satisfies

(5.2) $$Q(SA\nu) - Q(A\nu) = -(\mu + \tau)(1 - \mu)(b - \tfrac{1}{2})(\delta - \theta\gamma)^2$$

if the constants a and b satisfy

(5.3) $$a - 1/2 = \theta(b - 1/2); \quad \theta \equiv (1 - \tau)/(1 - \mu) .$$

Here,

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Indeed, if the left hand side of (5.2) maintains one sign for all ν then (5.2) and (5.3) necessarily hold. In the context of schemes in class N this observation yields the following result. Fix the local base sate u_s and put

$$A = A_j(u_s), \ \mu = \mu_j(u_s) \quad \text{and} \quad \tau = \tau_j(u_s) .$$

It follows from (3.7) and (3.8) that

$$(1 - \tau_j)/(1 - \mu_j) = \theta_j(u_s) \equiv \{\alpha_t + \alpha_x\lambda_j(u_s)\}/\{\beta_t + \beta_x\lambda_j(u_s)\} .$$

Thus, if we set

$$\phi_j(\alpha,u_\alpha) = b_j(u_s)$$

if α is a j-wave crossing the ws side of a mesh diamond with base state u_s and

$$\phi_j(\alpha,u_\alpha) = a_j(u_s)$$

if α is a j-wave crossing the se side of a mesh diamond with base state u_s then the functional $P = \sum_{j=1}^{n} P_j$ with P_j defined by (4.3) satisfies the inquality (4.2) if a_j and b_j satisfy

(5.4) $$(a_j - 1/2) = \theta_j(b_j - 1/2);\ b_j > 1/2 ,$$

and if TVX << 1. We note that changes in the arguments of the weights ϕ_j leads only to cubic contributions which can be absorbed into $e_3(X)$. We note that the restriction (5.4) depends only on the eigenvalues λ_j and the choice of normals α and β and leads to a one-parameter family of potentials P_j. This is in accordance with the fact that any finite difference scheme with smooth generating function consistent with the equations is at least first order accurate on smooth solutions: if a mesh diamond is embedded in an exact simple wave and if the values of the exact solution at the west, south and east vertices are substituted into the generating function ϕ, then the value produced by ϕ differs from the value of the exact solution at the north vertex by a quantity on the order of Δx^2.

The potential T(X) for transverse interactions is constructed as follows. We begin by considering a slightly more general potential of the form

(5.3) $$T(X) = s \sum_{A} |\alpha\beta| + w \sum_{R} |\alpha\beta| + \sum p(\alpha,\beta)|\alpha\beta| + \sum r(\alpha,\beta)|\alpha\beta| .$$

Here A(X) and R(X) denote respectively the sets of all approaching and receding pairs of waves in X such that each member wave is associated with a

different pair of adjacent mesh points; $p(\alpha,\beta) = p_{jk}(u_s)$ if α is a j-wave and β a k-wave crossing the ws side of a mesh diamond with base state u_s, $p = 0$ otherwise; $r(\alpha,\beta) = r_{jk}(u_s)$ if α is a j-wave and β a k-wave crossing the se side of a mesh diamond with base state u_s, $r = 0$ otherwise. We shall show that $T(X)$ satisfies (4.5) for an appropriate choice of coefficients, in particular, for choices such that s is constant, w vanishes and p_{jk} and r_{jk} are smooth positive functions of u_s. To this end we write the linear map (5.1) in the form

$$\alpha = \gamma + a(\delta - \theta\gamma)$$

$$\beta = \delta - a(\delta - \theta\gamma), \quad a = 1 - \mu ,$$

in order to display the deviation for the corresponding simple wave, $\delta = \theta\gamma$. Consider a discrete wave interaction associated with a mesh diamond having a local base state u_s. Fixing the value u_s, the incoming and outgoing waves $\nu = (\delta,\gamma)$ and $\sigma = (\alpha,\beta)$ are related as follows modulo a term on the order $|\nu|^2$:

$$\alpha_j = \gamma_j + \psi_j, \quad \psi_j \equiv a_j(\delta_j - \theta_j\gamma_j) , \tag{5.4}$$
$$\beta_j = \delta_j - \psi_j$$

where $a_j = a_j(u_s)$ and $\theta_j = \theta_j(u_s)$. In order to establish (4.5) it is sufficient to show that the following incoming and outgoing potentials associated with a single discrete wave interaction,

$$T_i(\nu) = s \sum \gamma_j\delta_k + w \sum \delta_j\gamma_k + \sum p_{jk}\delta_j\delta_k + \sum r_{jk}\gamma_j\gamma_k , \tag{5.5}$$

$$T_0(\sigma) = s \sum \beta_j\alpha_k + w \sum \alpha_j\beta_k + \sum p_{jk}\beta_j\beta_k + \sum r_{jk}\alpha_j\alpha_k , \tag{5.6}$$

with summations taken over indices $j < k$, satisfy

$$T_0(\sigma) - T_i(\nu) \leq -\text{const}\ \tau(\nu) + \text{const} \sum_{j<k} |\psi_j \psi_k|$$

(5.7)

$$\tau(\nu) \stackrel{\text{def}}{=} \sum_{j<k} |\nu_j||\nu_k|, \quad |\nu_\ell| = |\delta_\ell| + |\gamma_\ell|$$

provided that (5.4) holds and the coefficients are properly selected. We note that the quadratic correction terms to (5.4) as well the changes in the local base state introduce only admissible cubic terms into formulas (5.7). Lastly, we remark that without loss of generality one can restrict attention to the case where δ_j and γ_j are non-negative for $j = 1,2,\ldots,n$ since the general case follows by replacing δ_j and γ_j with $|\delta_j|$ and $|\gamma_j|$ respectively.

Next, we shall describe the calculations which lead to the restrictions on s,w,p_{jk} and r_{jk} which guarantee (5.5). In the following all summations are taken over indices $j < k$. Substituion of (5.4) into (5.6) yields

$$T_0(\sigma) - T_i(\nu) = (s - w) \sum \delta_j\gamma_k - \gamma_j\delta_k + \sum (r_{jk} + p_{jk} - s - w)\psi_j\psi_k$$
$$+ \sum (s\sigma_j - p_{jk}\delta_j - w\gamma_j + r_{jk}\gamma_j)\psi_k + \sum (w\delta_k - p_{jk}\delta_k - s\gamma_k + r_{jk}\gamma_k)\psi_j$$

A brief calculation show that

$$(1 + \theta_j)(1 + \theta_k)(\delta_j\gamma_k - \gamma_j\delta_k) = (\theta_j - \theta_k)(\delta_j + \gamma_j)(\delta_k + \gamma_k)$$
$$+ (1 + \theta_k)(\gamma_k + \delta_k)\psi_j/a_j - (1 + \theta_j)(\gamma_j + \delta_j)\psi_k/a_k ,$$

and therefore

$$T_0(\sigma) - T_i(\nu) = (s-w) \sum (\theta_j - \theta_k)(\delta_j + \gamma_j)(\delta_k + \gamma_k)/(1+\theta_j)(1+\theta_k)$$

(5.8)
$$+ \sum [\{w - p_{jk} + (s-w)/a_j(1+\theta_j)\}\delta_k + \{r_{jk} - s + (s-w)/a_j(1+\theta_j)\}\gamma_k]\psi_j$$
$$+ \sum \{s - p_{jk} + (s-w)/a_k(1+\theta_k)\}\delta_j + \{r_{jk} - w - (s-w)/a_k(1+\theta_k)\}\gamma_j]\psi_k$$

The condition that the coefficients of ψ_j and ψ_k in the second and third summations of (5.8) vanish when $\delta_j = \theta_j\gamma_j$ and $\delta_k = \theta_k\gamma_k$ respectively leads to the following pair of linear equations for r_{jk} and p_{jk} in terms of (specified) values of s and w

$$
\begin{aligned}
r_{jk} - \theta_k p_{jk} &= s - \theta_k w - (1 + \theta_k)\tau_{jk} \\
r_{jk} - \theta_j p_{jk} &= w - \theta_j s + (1 + \theta_j)\tau_{jk} ,
\end{aligned}
\tag{5.9}
$$

where $\tau_{jk} = (s - w)/a_j(1 + \theta_j)$. The system (5.9) has a unique solution since $\theta_j \neq \theta_k$ if $j \neq k$. In particular we note that if we take $w = 0$ and s to be a positive number then the solutions r_{jk} and p_{jk} are positive and we obtain an identity of the form

$$
\begin{aligned}
T_0(\sigma) - T_i(\nu) = s \sum\ & (\theta_j - \theta_k)(\delta_j + \gamma_j)(\delta_k + \gamma_k)/(1 + \theta_j)(1 + \theta_k) \\
& + \sum\ [s\{1/a_j(1 + \theta_j) - 1/a_k(1 + \theta_k) + 1\} - 2p_{jk}]\psi_j\psi_k ,
\end{aligned}
$$

in which the first summation involves a negative coefficient and the second is on the order of the square of the distance to the corresponding simple waves. It follows that the functional $T(X)$ of the form (5.3) satisfies the desired estimate (4.5) if we take $w = 0$, s positive and r_{jk} and p_{jk} as the solutions of (5.9). Here the functions r_{jk} and p_{jk} will depend smoothly on the local base state u_s since the equations (5.9) depend smoothly on u_s. We note that an analogous functions with $w \neq 0$ can be constructed but, as we shall see, such a functional is less convenient for the purposes of hybridization with the random choice method. Indeed, the act of attributing a potential for interacting of the form $w|\alpha\beta|$ to a pair of receding waves α and β in the random choice method leads to several unnecessary terms which are awkward to handle. These terms are simply avoided by setting $w = 0$.

6. REFLECTION AND CANCELLATION OF WAVES.

We observed in Section 4 that a functional of the form $TV + c_1P + c_2T$ is decreasing along these segments of the orbit M^kX with small total variation which lie in the complement of the weakly interacting states W_{m2}. In this

section we shall construct, for a given generator ϕ in class N, a quadratic perturbation

$$q = q(w,s,e) = O\{|w - s|^2 + |s - e|^2\} ,$$

with the property that the reflected waves produced by the generator $\phi + q$ on local states near simple waves are third order with respect to the incoming magnitudes and the property that the marching map M associated with $\phi + q$ satisfies

$$TVMX - TVX \leq const.\{\tau(X) + d^2(X) + e_3(X)\} ,$$

if osc X is small, i.e. in one time-step the total variation norm can not increase by more than a quantity on the order of the corresponding reduction in the potential $Q = c_1P + c_2T$, modulo a quantity on the order $e_3(X)$ of the approximate rate of entropy production. It follows that for appropriate m the functional $F = TV + Q$ is decreasing along those segments of the orbit with small total variation which lie in the complement of W_{m3} and that

$$F(MX) - F(X) \leq const.e_3(X)$$

for arbitrary configurations X with small total variation.

In this section we shall also describe the effective cancellation between shocks and rarefaction waves of the same field which exists for class N schemes and their perturbations and compare it with the corresponding cancellation occurring in the exact solution. To this end we shall begin by recalling that for schemes in N the outgoing and incoming magnitudes of a local interaction and related by formulas of the form

$$\begin{aligned} \alpha_j &= (1 - \mu_j)\delta_j + \tau_j\gamma_j + p_j(\nu); \quad p_j(\nu) = O(|\nu|^2) , \\ \beta_j &= \mu_j\delta_j + (1 - \tau_j)\gamma_j + q_j(\nu); \quad q_j(\nu) = O(|\nu|^2) , \end{aligned} \tag{6.1}$$

where $\nu = (\nu_1,\ldots,\nu_n)$ and the functions μ_j,τ_j,p_j and q_j depend smoothly on the base state of the associated mesh diamond. The process connecting the

incoming and outgoing magnitudes $\nu_j = (\delta_j, \gamma_j)$ and $\sigma_j = (\alpha_j, \beta_j)$ can be viewed as a composition of binary interactions by writting (6.1) in the form

$$\alpha_j = \alpha_j(\nu) = \sum_{k=1}^{n} \alpha_j(\hat{\nu}_k) + O\{\tau(\nu)\}$$

(6.2)

$$\beta_j = \beta_j(\nu) = \sum_{k=1}^{n} \beta_j(\hat{\nu}_k) + O\{\tau(\nu)\} \ .$$

Here $\hat{\nu}_k \in R^n$ represents the vector whose k^{th} component equals ν_k and whose j^{th} component $j \neq k$ vanishes; the local base state is regarded as fixed and $\tau(\nu)$ measures the total transverse wave interaction:

$$\tau(\nu) = \sum_{j<k} |\nu_j||\nu_k| \ .$$

Indeed, it is easy to show that a formula of the form (6.2) holds for an arbitrary function $\alpha = \alpha(\nu)$ vanishing at the origin and having a bounded second derivative.

We shall construct the perturbation q in such a way that incoming k-waves, produce outgoing j-waves, $j \neq k$, from the generator $\phi + q$ which are on order of the distance squared to the corresponding k-simple waves modulo a cubic error and in such a way that conservation of wave magnitudes holds in the j^{th} field modulo a cubic error, i.e.

$$|\alpha_j(\hat{\nu}_k)| + |\beta_j(\hat{\nu}_k)| \leq \text{const.}|\delta_j - \theta_k(u_s)\gamma_k|^2 + \text{const.}|\nu_k|^3 \ ,$$

$$\alpha_j(\hat{\nu}_j) + \beta_j(\hat{\nu}_j) = \delta_j + \gamma_j + O(|\nu_j|^3) \ .$$

It follows that (6.2) can be written in the convenient form

$$\alpha_j = (1 - \mu_j)\delta_j + \tau_j\gamma_j + p_j(\hat{\nu}_j) + O\{g(X)\}$$
$$\beta_j = \mu_j\delta_j + (1 - \tau_j)\gamma_j + q_j(\hat{\nu}_j) + O\{g(X)\}$$

where

$$p_j(\hat{\nu}_j) + q_j(\hat{\nu}_j) = O(|\hat{\nu}_j|^3)$$

$$g(X) = \tau(X) + d^2(X) + |\nu|^3$$

and $X = (\delta,\gamma,u_s) \in R^{3n}$ specifies the incoming configuration. One can then employ (6.3) to display the effective cancellation experienced by interacting *j-shocks and j-rarefaction waves by noting that*

$$|\alpha_j| + |\beta_j| \leq |\delta_j| + |\gamma_j| - C_j(\nu) + O\{g(X)\}$$

where

$$C_j(\nu) = C(\delta_j,\gamma_j) - C(\alpha_j,\beta_j)$$

$$C(x,y) = |x| + |y| - |x + y| = \left\{ \begin{array}{ll} 2\min(|x|,|y|) & \text{if } \operatorname{sgn} x \neq \operatorname{sgn} y \\ 0 & \text{otherwise} \end{array} \right\} .$$

Furthermore, it is not difficult to show that the effective cancellation $C_j(\nu)$ in the j^{th} field is bounded below by a fraction of the cancellation $C(\delta_j,\gamma_j)$ which occurs in the interaction of waves δ_j and γ_j in the random choice method, modulo a quadratic term:

$$C_j(\nu) \geq k_j C(\delta_j,\gamma_j) - O\{g(X)\}$$

$$k_j = \frac{1}{2} \min(\mu_j, 1 - \mu_j, \tau_j, 1 - \tau_j) .$$

Thus, we obtain the following estimates on the outgoing waves of local and global interactions:

$$|\alpha_j| + |\beta_j| \leq |\delta_j| + |\gamma_j| - k_j C(\delta_j,\gamma_j) + \text{const.} g\{(\delta,\gamma,u_s)\}$$

$$TVMX - TVX \leq -\text{const.} C(X) + \text{const.}\{d^2(X) + \tau(X) + e_3(X)\}$$

where the cancellation in a global configuration $X = \{(\delta^k,\gamma^k,s_k)\}$ is defined by

$$C(x) = \sum_{j,k} C(\delta_j^k,\gamma_j^k) .$$

In contrast, the interaction of a j-shock and j-rarefaction wave in the random choice method leads to the absorption of the smaller wave by the larger and a concurrent decrease in the total variation norm equal to twice the magnitude of the smaller wave modulo quadratic terms: the outgoing j-wave ε_j is related to the incoming j-waves δ_j and γ_j by a formula of the form

$$|\varepsilon_j| = |\delta_j| + |\gamma_j| - C(\delta_j,\gamma_j) + O(|\nu|^2)$$

cf. [13]. For conservative difference schemes the presence of fractional cancellation leads to the persistance of small oscillations, corresponding to alternating sequences of shocks and rarefactions, over several time steps.

We shall next construct an appropriate perturbating q for a given generator ϕ in class N. Consider a discrete interaction associated with a mesh diamond and regard the base state u_s as fixed. Suppose that the incoming waves consist of just a pair of j-rarefaction waves, i.e. suppose $\delta_j \geq 0$, $\gamma_j \geq 0$ and $\nu_k = 0$, $k \neq j$. Here the values at the west and south vertices u_w and u_s lie on the j-wave curve $\Gamma_j(s)$ through u_s, the integral curve in R^n of the right eigenvector $r_j(u)$, and satisfy

$$\lambda_j(u_w) \leq \lambda_j(u_s) \leq \lambda_j(u_e) .$$

If the generator ϕ were to produce a point on Γ_j then reflected waves are absent:

$$\alpha_j + \beta_j = \delta_j + \gamma_j; \quad \alpha_k,\beta_k = 0, \ k \neq j .$$

However, in general, the point $n_j = n_j(\delta_j,\gamma_j,s)$ produced by ϕ for configurations with $\nu_k = 0$, $k \neq j$ lies within a distance on the order of $|\nu_j|^2$ from the nearest point of $\Gamma_j(s)$ which we shall denote by

$$m_j = m_j(\delta_j,\gamma_j,s) .$$

We shall first construct a perturbation q_j which reduces to $m_j - n_j$ if ν_j forms a discrete j-rarefaction wave and vanishes if

$$|\delta_j - \theta_j(u_s)\gamma_j| > \text{const.}(|\delta_j| + |\gamma_j|) .$$

Fix $m > 0$, let $z(x)$ be a smooth even function equal to one of $x < m$ and equal to zero if $x > 2m$ and put

$$q_j(\nu) = q_j(\nu_j) \equiv z\{(\delta_j - \theta_j(u_s)\gamma_j)/(\delta_j + \gamma_j)\}\{m_j - n_j\} .$$

The desired perturbation q is obtained by taking m small and defining

$$(6.4) \qquad q = \sum_{j=1}^{n} d_j(\nu)q_j(\nu); \quad d_j = \prod_{k\neq j} z\{(\delta_k^2 + \gamma_k^2)/(\delta_j^2 + \gamma_j^2)\} .$$

We note that while the formula (6.4) for q is motivated by the case where the incoming configuration consists of just a pair of incoming j-rarefaction waves it has the desired effect for general configurations. In particular we note that the support of q is contained within the set of all states $(\delta,\gamma,u_s) \in R^{3n}$ with the property that there exists an index, say j, such that

$$|\nu_k| \leq 4m|\nu_j|, \quad k \neq j$$

$$|\delta_j - \theta_j(u_s)\gamma_j| \leq 2m(|\delta_j| + |\gamma_j|); \text{ sgn } \delta_j = \text{sgn } \gamma_j .$$

Furthermore a straight forward calculation shows that q vanishes together with its first derivative in ν at $\nu = 0$ and has a bounded second derivative: roughly speaking q makes a change on the order of ε^2 over a distance on the order of ε.

7. THE RANDOM CHOICE SCHEME.

We shall briefly describe the generating function of the random choice method together and the potential functionals which are used to obtain a uniform estimate on the total variation norm of the corresponding difference approximations. We shall compare these functionals with the functionals

constructed in Section 5 for schemes in class N and describe certain modifications of the random choice generating function which facilitate a hybridization with schemes in class N.

The random choice method can be based on a centered diamond-shaped stencil as follows. Let $R(x/t;u^-,u^+)$ denote the classical solution of the Riemann problem with data (u^-,u^+) [20], and let (y_k) be a sequence equidistributed in the interval $(-1,1)$. Consider a grid whose stencil has vertices of the form

$$n = \{x,(m + 1)\Delta t\} \qquad w = \{x - \Delta x,m\Delta t\}$$

$$s = \{x,(m - 1)\Delta t\} \qquad e = \{x + \Delta x,m\Delta t\}$$

where x is an integral multiple of Δx. The value of the random choice approximation at the north vertex depends only on the values at the west and east vertices and the corresponding element y_m; it is obtained by solving the Riemann problem with data (u_w,u_e) and sampling the value of the solution at time $t = \Delta t$ and position $x = y_m\Delta x$, i.e.

$$u_n = u_n(u_w,u_e,y_m) = R(y_m\Delta x/\Delta t;u_w,u_e) ,$$

the C-F-L condition is enforced in the standard fashion. For convenience one can associate with each grid function produced by the random choice method a piecewise constant function which assumes, on each small parallelogram centered at a mesh point, the corresponding value of the grid function. We note that the random choice method in its original formulation [12] involves approximate solutions which are exact in strips of the form $m\Delta t < t < (m + 1)\Delta t$. However, it is not difficult to show that if a sequence of such approximate solutions is stable in the total variation norm and convergent pointwise a.e. then the corresponding piecewise constant functions described above converge pointwise a.e. to the same limit.

In order to discuss the functionals introduced by Glimm for the random choice method, we shall briefly recall the laws governing the local wave

interactions therein. We first note that the outgoing wave magnitudes of a local interaction associated with a single mesh diamond have the following structure. Let $\varepsilon_j = \varepsilon_j(u_w,u_e)$ denote the magnitudes of the waves in the Riemann problem with data (u_w,u_e). There exists an index m such that

$$\begin{aligned} &\alpha_j = \varepsilon_j, \quad \beta_j = 0 \ ; \quad j = 1,\ldots,m-1 \\ &\alpha_j = 0 \ , \quad \beta_j = \varepsilon_j; \quad j = m+1,\ldots,n \end{aligned} \tag{7.2}$$

Furthermore, if $\varepsilon_m < 0$ then either $\alpha_m = \varepsilon_m$ and $\beta_m = 0$ or $\alpha_m = 0$ and $\beta_m = \varepsilon_m$ accordingly as the sample point $y_m\Delta x/\Delta t$ lies to the right or left of the m-shock of the Riemann problem. If $\varepsilon_m > 0$ then either the aforementioned relation holds or the sample point splits the m-rarefaction wave of the Riemann problem and produces waves $\alpha_m > 0$, $\beta_m > 0$ which satisfy $\alpha_m + \beta_m = \varepsilon_m$. In contrast to conservative difference schemes, local interactions in the random choice method can only increase the number of waves (by one) if a rarefaction wave is split and such splitting is not accompanies by wave amplification in the sense that

$$\begin{aligned} &|\varepsilon_j| = |\delta_j| + |\gamma_j| - C(\delta_j,\gamma_j) + O\{D(\delta,\gamma)\} \\ &D(\delta,\gamma) = \sum \{|\delta_k\gamma_j| : \delta_k \text{ and } \gamma_j \text{ approach}\} \ , \end{aligned} \tag{7.3}$$

in particular, two rarefaction waves of the same field do not approach, cf. Section 5. It then follows from (7.3), together with the local recession of waves after interaction as expressed by (7.2), that the functional

$$Q(X) = \sum \{|\alpha\beta| : \alpha,\beta \text{ approaching in } X\}$$

compensates for wave amplification, i.e.

$$\begin{aligned} &Q(MX) - Q(X) \leq -\text{const.} \sum D(\delta^k,\delta^k) \\ &F(MX) - F(X) \leq 0; \ F \overset{\text{def}}{=} TV + \text{const.}\ Q \ , \end{aligned}$$

if $X = \{(\delta^k, \gamma^k, s_k)\}$ has small total variation [12]. We note that $F(x)$ is independent of the base states of X, their influence is absorbed into the last term on the right hand side of (7.3).

The local recession of waves also plays a central role in our hybridized methods. In this connection we shall next remark on several ways in which the structure of the equidistributed sequence (y_k) influences the geometry of interacting waves and we shall present two modifications of the random choice generator which guarantees that the local recession of waves occurs in a minimal number of time steps. For simplicity let us restrict our attention to systems of two equations. Consider a corresponding exact solution which consists of two interacting shocks of different fields, cf. the solution $\tilde{u}$, discussed in the subsection on self-interactions in Section 3. For a solution of the form $\tilde{u}$, the random choice approximations consist of exactly two shocks in each strip $m\Delta t < t < (m + 1)\Delta t$; for a typical equidistributed sequence the shocks approach during a time interval of the form $(0,p\Delta t)$, interact at $t = p\Delta t$, remain adjacent in $(p\Delta t,q\Delta t)$ and separate in $(q\Delta t,\infty)$. The shocks remain adjacent in $(p\Delta t,q\Delta t)$ if the sample point lies strictly to the right or left of the waves in the associated Riemann problem; of course, since the sequence is equidistributed this can not happen for an arbitrarily long string of its elements, hence the waves eventually separate.

Certain technical problems associated with delayed wave recession in hybridized schemes can be avoided by using an equidistributed sequence such that at least one element in every string of m consecutive elements corresponds to a point in the wake region between characteristic fields. In order to describe the construction of such sequences let us restrict our attention, for concreteness, to systems with symmetric eigenvalues, i.e. $\lambda_1(u) = -\lambda_2(u)$; the general case is handled analogously. We note that systems of the form (1.4)

have symmetric eigenvalues. Let us suppose that all analysis takes place in the neighborhood $N(\bar{u})$ of a fixed vector $\bar{u} \in R^n$ and write the interval $I = (-1,1)$ in the form

$$I = I_1 \cup I_2 \cup I_3$$

where $I_1 = (-1,-1/2]$, $I_2 = (-1/2,1/2)$ and $I_3 = [1/2,1)$. Consider an equidistributed sequence (y_k) in I which satisfies

(7.4) $y_{2k} \in I_2$ for all k; $y_{2k+1} \in I_1$ for k even; $y_{2k+1} \in I_3$ for k odd.

Such sequences can be constructed, for example, by starting with a sequence z_k equidistributed in $(0,1)$, scaling and translating in the obvious way to obtain three sequences (a_k), (b_k) and (c_k) equidistributed in I_1, I_2, and I_3 and then arranging their elements in the alternating manner indicated by (7.4), i.e.

$$y_{4k} = b_k;\ y_{4k+1} = c_k;\ y_{4k+2} = b_{k+1},\ y_{4k+3} = a_k .$$

Now, if the C-F-L number is chosen so that

$$\{\lambda_1(y)\Delta t/\Delta x : u \in N(\bar{u})\} \subset I_1 \quad \text{and} \quad \{\lambda_2(u)\Delta t/\Delta x : u \in N(\bar{u})\} \quad I_3$$

then every second member of the sequence (y_k) corresponds to a sample point $y_k \Delta t/\Delta x$ which lies in the wake region in the sense that

$$\max_{N(\bar{u})} \lambda_1 < y_k \Delta t/\Delta x < \min_{N(\bar{u})} \lambda_2 .$$

In working in the context of general systems or in the absence of any restrictions on the C-F-L number (except that it be less than one) one will obtain an equidistributed sequence such that every m^{th} element $m > 2$ lies in the wake region.

We note that the process above which construct y_k from z_k does not alter the error of approximation of z_k in the following sense. If the average number of elements of z_k with index $\leq n$ in a given subinterval J

approximates half the length of J within an error on the order of $\psi(n)$, i.e.

$$|S\{n,j;(z_k)\}/n - |J|/2| \leq c(J)\psi(n)$$

where, without loss of generality $\psi(n)$ vanishes monotonically, then the same rate of approximate obtains for (y_k), i.e.

$$|S\{n,J;(y_k)\}/n - |J|/2| \leq d(J)\psi(n)$$

for an appropriate constant $d(J)$ depending only interval J. For a typical equidistributed sequence one has $\psi(n) = n^{-1/2}$, while quadratic irrationalities lead to sequences for which $\psi(n) = \log n/n$. For technical reasons, concerning hybridized schemes, it will be necessary for us to employ equidistributed sequences with a rate $\psi(n)$ satisfying

$$\lim_{n\to\infty} \psi(n)n^{3/4} = 0 \tag{7.5}$$

in order to show that limit of the associated difference approximations is an exact solution. As we shall see in Section 11 the restrictin (7.5) is not necessary for the stability of the hybridized schemes.

Next, we shall describe a second modification of the random choice generator which has the desirable effect of emphasizing the local recession of waves after interaction. For this purpose, let us consider a random choice interaction for a system of two equations and suppose that the sample point $y_k\Delta x/\Delta t,\Delta t)$ does not lie between the waves ε_j in the solution of the Riemann problem with data (u_w,u_e) obtained from the west and east vertices of the associated mesh diamond. In this situation one of four configurations obtains for the outgoing waves (α,β): both cross the wn side; both cross the ne side; ε_1 is a rarefaction wave split by y_k, i.e.,

$$\alpha_1 > 0,\ \alpha_2 = 0,\ \beta_1 > 0,\ \alpha_1 + \beta_1 = \varepsilon_1\ ;$$

ε_2 is a rarefaction wave split by y_k, i.e.,

$$\alpha_2 > 0,\ \beta_2 > 0,\ \beta_1 = 0,\ \alpha_2 + \beta_2 = \varepsilon_2 .$$

If any of the above four configurations occur and if

$$|\varepsilon_1\varepsilon_2| > \varepsilon\sigma^2,\ |\varepsilon_1| + |\varepsilon_2| < L\sigma \tag{7.6}$$

with ε and L given positive constants, we replace the original element y_k by zero. Thus, in the case of outgoing waves whose magnitudes are, so to speak, of order σ we enforce the immediate recession of waves if it does not occur in the original choice of equidistributed sequence.

The effect of such a modification on the error associated with the random choice difference approximations can be analyzed as follows. Let $\Omega(i,j)$ denote the mesh diamond centered at $(i\Delta x, j\Delta t)$. The random choice approximations $u = u(x,t;\Delta x,\Delta t)$ satisfy the exact equations (1.1) within an error of the form

$$\sum_{(i,j)} E(i,j) = \iint \phi_t u + \phi_x f(u)\,dx\,dt \tag{7.7}$$

where

$$\begin{aligned} E(i,j) &= \int_{-\Delta x}^{\Delta x} \phi(x,(j+1)\Delta t)\{u_n - R(x/\Delta t;u_w,u_e)\}dx \\ u_n &\equiv R(y_{j+1}\Delta x/\Delta t;u_w,u_e) \end{aligned} \tag{7.8}$$

It has been shown by Liu [26] that for each equidistributed sequence the sum of all the errors $E(i,j)$ associated with mesh diamonds $\Omega(i,j)$ vanishes for each given test function ϕ as the mesh length approaches zero. The additional deviation produced by the above relocation of certain elements y_k is on the order $\Delta x(|\varepsilon_1| + |\varepsilon_2|)$ and satisfies

$$\Delta x(|\varepsilon_1| + |\varepsilon_2|) < \Delta x L|\varepsilon_1\varepsilon_2|/\varepsilon\sigma$$

It follows that the sum of all such terms vanishes as Δx approaches zero provided that $\sigma(\Delta x)$ satisfies

$$\lim_{\Delta x \to 0} \Delta x/\sigma(\Delta x) = 0 \tag{7.9}$$

Here we have appealed to the fact that the total amount of wave interaction as

measured by D in a given random choice approximation is bounded uniformly with respect to the mesh length [12, 13], which in this context implies

$$\sum |\varepsilon_1 \varepsilon_2| \leq \text{const.} ,$$

where the summation is taken over all diamonds associated with a relocated element y_k.

In treating systems of two equations with eigenvalues of the form $\lambda_2 = -\lambda_2$, we shall, in the remaining sections, refer to the random choice generator which employs an equidistributed sequence satisfying (7.4) and subject to the above alteration based on (7.6) as the modified random choice generator.

8. HYBRIDIZED SCHEMES.

In this section we shall describe a class of hybridized methods based on the tracking of waves whose magnitude lies between specified thresholds depending on the mesh length. Let us begin by considering two generating functions based on the same grid having a centered diamond-shaped stencil,

$$u_n = \phi(u_w, u_s, u_e) \tag{8.1}$$

$$u_n = r(u_w, u_e) \tag{8.2}$$

where ϕ corresponds to a (possibly perturbed) scheme in class N and r to the random choice method; the dependence of r on the equidistributed sequence is suppressed. Suppose one is given a mesh function v which satisfies either (8.1) or (8.2) at each mesh diamond, i.e. the value at the north vertex of each mesh diamond is obtained from the three lower vertices by either (8.1) or (8.2) according to some presently unspecified rule of selection. Let C and RC denote the sets of diamonds at which (8.1) and (8.2) are employed and consider the associated pattern of wave magnitudes in v as introduced in Section 3. For a given mesh diamond Ω let Ω^- and Ω^+ denote respectively the diamonds whose ne side coincides with the ws side of Ω and whose wn side

coincides with the se side of Ω; thus an incoming wave of Ω crossing its ws side serves as an outgoing wave of Ω^- while an incoming wave crossing its se side serves as an outgoing wave of Ω^+. Suppose one is given constants $m > k > 1$ and a positive function $\sigma = \sigma(\Delta x)$ which vanishes in the limit as Δx approaches zero. In terms of these parameters we define a <u>major j-wave</u> in v to be an order tuple

$$(\varepsilon_j^1, \varepsilon_j^2, \ldots, \varepsilon_j^p)$$

whose elements ε_j^k denote the magnitude of an incoming j-wave of some diamond in RC, say Ω_k and satisfy the following conditions.

(8.3) If ε_j^1 is the outgoing wave of a diamond in C then $|\varepsilon_j^1| > m\sigma$. If ε_j^1 is the outgoing wave of a diamond in RC then $|\varepsilon_j^1| > k\sigma$ and Ω_1 contains no incoming j-wave δ_j such that

$$\text{sgn } \delta_j = \text{sgn } \varepsilon_j^1 \quad \text{and} \quad |\delta_j| > k\sigma .$$

(8.4) If $p > 1$ then ε_j^k and ε_j^{k+1} serves as the incoming and outgoing j-waves of Ω_k and satisfy

$$\text{sgn } \varepsilon_j^k = \text{sgn } \varepsilon_j^{k+1} \quad \text{and} \quad |\varepsilon_j^k| > \sigma .$$

We shall say that a mesh diamond Ω contains an <u>incoming major j-wave</u> if any of the following five conditions hold.

1) $|\delta_j| > m\sigma$ and $\Omega^- \in D$ 2) $|\gamma_j| > m\sigma$ and $\Omega^+ \in D$

3) $|\delta_j| > k\sigma$ and $\Omega^- \in RC$ 4) $|\gamma_j| > k\sigma$ and $\Omega^+ \in RC$

5) Either δ_j or γ_j lies on a major j-wave.

We note that if any of the first four conditions hold then it is possible for Ω to serve as the initial diamond of a major j-wave.

In this paper we shall establish the stability and convergence of certain hybridized schemes whose switching operators are based on the tracking of major waves. The starting values on the first two time-levels $t = 0$ and $t = \Delta t$ for such schemes can be obtained by either setting

$$u(j\Delta x,0) = u_0(j\Delta x); \quad u(k\Delta x,\Delta t) = u_0(k\Delta x)$$

for appropriate indices j and k or by using standard procedures to produce more accurate values at the second level $t = \Delta t$ from the initial data $u_0(x)$. Given a selection of parameters $m > k > 1$ and $\sigma = \sigma(\Delta x) > 0$ together with starting values on the first two levels, the difference approximation is marched forward by using (8.1) if Ω contains an incoming major j-wave for some index $j = 1,2,\ldots,n$ while (8.2) is used otherwise. For a class of systems of two equations and initial data with small total variation we shall establish in Section 11 convergence of certain hybridized schemes of the above type which employ perturbations of class N generators for the function ϕ in (8.1).

It would also be interesting to study hybridized methods based on a single-level threshold where the random choice generator is used to compute the value at the north vertex of a mesh diamond if there exists an incoming wave δ_j or γ_j such that $|\delta_j| \geq \sigma$ or $|\gamma_j| \geq \sigma$, while a conservative generator ϕ is used otherwise. Such schemes correspond to more standard generating functions of the form

$$\theta\phi + (1 - \theta)r$$

for an appropriate switching function θ. The reason for introducing a multi-level threshold in this paper is to ensure the convergence of the corresponding difference approximations to an exact solution. Each difference approximation produced by hybridizations of the above type satisfies the exact equations (1.1) within an error which consists of three parts: the first is associated with diamonds in C and vanishes by the usual arguments employed for standard schemes [22], the second is associated with the equidistributed sequence on diamonds in RC and vanishes by the argument of Liu [26], and the third involves the total magnitude of waves crossing the boundary between C and RC. In Section 11, we shall show that the third contribution vanishes as Δx

approaches zero. To do this, it seems necessary to rule out rapid switching between the two generators (8.1) and (8.2) during the propagation of an individual wave. In hybridizations of the above type, the random choice generator remains locked on a major wave until a measurable amount of interaction on the order of $\sigma(\Delta x)$ has reduced its magnitude below σ. The expectation that such methods will produce only a finite amount of "wave interaction" similar to that of the exact solution operator leads one to suspect that the boundary contribution will vanish in the limit. The estimates which make these remarks precise are presented in Section 11. In this connection we also recall from Section 7 that we must require $\sigma(\Delta z)$ to vanish slower than Δx in order that our modification of the equidistributed sequence on selected random choice diamonds induces a deviation from the standard random choice error which vanishes in the limit. This restriction also prevents a premature switch from (8.1) to (8.2) in the computation of a focusing compression wave: if σ has the same order as Δx then the random choice generator is engage at a time on the order of Δt prior to the time of focus.

Finally, we remark that it would be interesting to study the corresponding hybridized methods which are based on incoming major j-waves which are of shock type. For technical reasons we have enlarged the class of incoming major j-wave to include both shocks and rarefaction waves. We conjecture however that as the mesh is refined the random choice method is in fact only engaged for a substantial length of time on the major shock waves.

9. HYBRIDIZED FUNCTIONALS.

In this section we shall construct potential functinals which will be used to estimate the total norm for the hybridized schemes of Section 7. To this end we first eliminate from $P_j(X)$ the terms associated with pairs of j-rarefaction

waves, since they are not compatible with the random choice potential which involves only pairs of approaching waves. We redefine the weights $\phi_j(\alpha,u_\alpha)$ of (5.3) by setting

(9.1) $$\phi_j(\alpha,u_\alpha) = b_j(u_\alpha) \quad \text{or} \quad \phi_j(\alpha,u_\alpha) = b_j(u_\alpha) - 1/2$$

if α is a j-shock wave or a j-rarefaction wave respectively, crossing the ws side of a mesh diamond with base state u_α and by setting

(9.2) $$\phi_j(\alpha,u_\alpha) = a_j(u_\alpha) \quad \text{or} \quad \phi_j(\alpha,u_\alpha) = a_j(u_\alpha) - 1/2$$

if α is a j-shock wave or a j-rarefaction wave respectively, crossing the se side of a mesh diamond with base state u_α. We then put

(9.3) $$P_j(X) = \sum_{A_j} |\alpha\beta| + \sum \phi_j(\alpha,u_\alpha)\alpha^2 ,$$

where $A_j = A_j(X)$ denotes the set of all pairs of approaching j-waves in X. As before the functions a_j and b_j are chosen to satisfy (5.4) and we define

$$P(X) = \sum_{j=1}^{n} P_j(X) .$$

The behavior of such functionals on local wave interactions can be efficiently described in terms of the connected polygonal arcs which consist of line segments joining adjacent mesh points. Following the standard convention [12], we shall refer to such an arc as an I-curve if the x-component varies monotonically and write $j_2 > J_1$ if J_2 lies toward larger time. Two I-curves $J_2 > J_1$ are called <u>consecutive</u> if they coincide except along the boundary of a mesh diamond Ω : $J_2 - J_1$ consists of the wn and ne sides of Ω while $J_1 - J_2$ consists of the ws and se sides of Ω. Here it is natural to associate with an I-curve J of a given mesh function u, the global configuration

$$X = X(J) = \{(\delta^k,\gamma^k,u_k)\}$$

which employs the values of a set of alternating vertices p_k of J together with the corresponding magnitudes of interpolated waves δ^k and γ^k. In the case of consecutive curves $J_2 > J_1$ the alternating vertices q_j of J_2 satisfy

$$q_j = p_j \quad \text{if} \quad j < k - 1 \quad \text{or if} \quad j > k + 1$$

$$q_k = n$$

where n denotes the north vertex of the separating diamond Ω. Figure 1 of Section 4 can be used to illustrate two consecutive I-curves $J_2 > J_1$ as follows: J_1 passes through the symbols s_{k-1}, u_k, s_k, u_{k+1}, s_{k+1} while J_2 passes through the symbols s_{k-1}, u_k, w_k, u_{k+1}, s_{k+1}. Here J_1 and J_2 are separated by the diamond Ω whose vertices are labelled by the symbols u_k, s_k, u_{k+1}, w_k. The alternating vertices of J_1 are labelled by the symbols s_k while the alternating vertices of J_2 contain the sequence s_{k-2}, s_{k-1}, w_k, s_{k+1}, s_{k-2}.

It follows easily from the results of Section 7 that the new functional P satisfies

$$P(J_2) - P(J_1) \leq - cd^2(\nu) + c\ \mathrm{osc}(J_1)C(\nu) + c|\nu|^3 \tag{9.4}$$

if $TVJ_1 << 1$ and if J_1 and J_2 are two consecutive I-curves separated by a mesh diamond with incoming waves $\nu = (\nu_1,\ldots,\nu_n)$, $\nu_j = (\delta_j,\gamma_j)$. Here we put

$$d^2(\nu) = \sum_{j=1}^{n} (\delta_j - \theta_j(u_s)\gamma_j)^2 \quad \text{and} \quad C(\nu) = \sum_{j=1}^{n} C(\delta_j,\gamma_j) .$$

As before the letter c denotes a positive constant depending only on the equations, the scheme and the state $\tilde{u}$ in R^n in the neighborhood of which all analysis takes place. The proof of (9.4) consists of the observation that the term 1/2 introduced in (9.1) and (9.2) accounts for the previous pairs of

j-rarefaction waves, while pairs with opposite signs introduce only a quantity on the order of the oscillation times the cancellation.

The weights employed in the transverse potential do not require modification, so we have

$$T(J_2) - T(J_1) \leq - c\tau(n) + cd^2(\nu) + c|\nu|^3 ,$$

for the perturbations of class N schemes introduced in Section 6, it follows that, for approrpriate constants, the functional $\tilde{F} = TV + cP + cT$ satisfies

$$\tilde{F}(J_2) - \tilde{F}(J_1) \leq - c\{\tau(\nu) + d^2(\nu) + C(\nu)\} + c|\nu|^3$$

if $TVJ_1 << 1$.

Appropriate functionals for the hybridized schemes introduced in Section 8 can be constructed by switching the weights ϕ_j and ψ of T and P on and off according to the following rule. Associate with an interpolated wave α the unique mesh diamond Ω for which α is outgoing. If $\Omega \in C$, let $\phi_j(\alpha,u_\alpha)$ keep its original value as defined by (9.1) and (9.2), otherwise put $\phi_j = 0$. If α and β cross the same side of Ω and $\Omega \in C$ let $\psi(\alpha,\beta)$ keep its original value, cf. Sections 4 and 5, otherwise put $\psi = 0$. Thus, the weights ϕ_j and ψ vanish if and only if the associated waves are outgoing for a random choice diamond. Without confusion, we shall employ the notation P_j and T for the hybridized functionals with aforementioned truncated weights.

The stability proof for the hybridized functionals given in Section 10 is motivated in part by several facts concerning the relative sizes of the weights associated with the physical and numerical potentials for schemes in class N. These facts, which we shall present below, are of independent interest and hopefully will be useful in future investigations. Let us now restrict attention to systems of two equations with eigenvalues λ_1 and λ_2 satisfying (1.2) and the symmetry condition $\lambda_1(u) = -\lambda_2(u)$. The well-known system

$$w_t + p(v)_x = 0, \quad v_t - w_x = 0$$

arising in fluid dynamics and elasticity provides an example of symmetric, genuinely nonlinear eigenvalues under the conditions $p' < 0$, $p'' > 0$. More, generally we note that systems of the form (1.4) have symmetric eigenvalues $\lambda_1 \equiv -\lambda_2$. For such systems it is natural to use class N schemes which are <u>symmetric</u> in the sense that they employ a centered stencil, $\alpha_t = \beta_t$, $\alpha_x = -\beta_x$ and functions $w_j = w_j(a)$ which satisfy $w_1 \equiv w_2$, cf. Section 3. We recall that the Lax-Friedrichs scheme is symmetric with $w_1 = w_2 \equiv 0$. A brief calculation show that for such symmetric systems the weights $r = r_{12}$ and $p = p_{12}$ associated with transverse group interactions coincide and exceed the weight associated with approaching transverse wave, i.e.

$$r = p > s .$$

We recall that if $w = 0$, r and p are uniquely determined by s and the local base state according to formulas (5.9). For such symmetric systems we also obtain a simplified formula discribing the effect of a local interaction onthe transverse potential T:

$$T(J_2) - T(J_1) \leq -s|\nu_1||\nu_2|(\theta_1 - \theta_2)/(1 + \theta_1)(1 + \theta_2) + (s - 2p)\psi_1\psi_2 + c|\nu|^3 ,$$

where $\psi_j = \delta_j - \theta_j(u_s)\gamma_j$. The formula (9.5) is useful in connection with the problem of forming a linear combination of P and T which is decreasing on the complement of the second order weakly interacting states W_2. To this end one might proceed by tryin, to dominate the indefinite term of T,

(9.6) $$(s - 2p)\psi_1\psi_2 ,$$

with the corresponding negative definite terms by which the functional P is reduced, i.e. the terms under the summation sign in the right hand side of the following inequality.

$$P(J_2) - P(J_1) \leq -\sum_{j=1}^{2} (\mu_j + \tau_j)(1 - \mu_j)(b_j - 1/2)\psi_j^2 + c(\text{osc } J_1)C(\nu) + c|\nu|^3 . \tag{9.7}$$

The formulas (9.5) and (9.7) suggest that one determine the restrictions on the weights s,p and b_j of symmetric schemes which imply that the following quadratic form is positive definite:

$$\sum_{j=1}^{2} (\mu_j + \tau_j)(1 - \mu_j)(b_j - 1/2)\psi_j^2 + (s - 2p)\psi_1\psi_2 . \tag{9.8}$$

Here, all of the coefficients are evaluated at the corresponding local base state and the values of ψ_j are regarded as arbitrary. Now for a fixed choice of s, the form (9.8) is clearly non-negative for appropriate choices of b_j. However, a brief calculation show that a positive definite form can not be achieved with a weights satisfying

$$p/2 \geq a_1 - 1/2, \quad p/2 \geq (b_2 - 1/2) . \tag{9.9}$$

This fact is of particular interest in connection with hybridized schemes since the quantities $a_1 - 1/2$ and $b_2 - 1/2$ represent the maximal weights associated with the numerical self-interactions of rarefaction waves as registered by P_j; we have

$$a_1 > b_1 \quad \text{and} \quad b_2 > a_1$$

since (5.4) holds and $\theta_1 < 1 < \theta_2$. As we shall see in Section 10 the numerical group interactions characterized by coefficients p and r generally lead to favorable contributions in bounding the hybridized functionals and it is natural to inquire into the extent to which they might compensate for the less favorable effects of switching on and off the coefficients associated with self-interactions.

Lastly, we remark that for the purpose of dominating the indefinite term (9.6) of T, one might appeal to the favorable term with leading coefficient -s on the right hand side of (9.5) and ask what restrictions on the weights s,p and b_j of symmetric schemes are implied by the condition that the following form be positive definite:

$$\sum (\mu_j + \tau_j)(1 - \mu_j)(b_j - 1/2)\psi_j^2$$

$$+ s(\theta_2 - \theta_1 - \varepsilon)|\nu_1||\nu_2|/(1 + \theta_1)(1 + \theta_2) + (s - 2p)\psi_1\psi_2 \ . \tag{9.10}$$

Here the quantity $\varepsilon > 0$ is introduced simply to guarantee that a residual term of order $\tau(\nu)$ is available to compensate for the corresponding growth in the total variation norm. A brief calculation shows that for that Lax-Friedrichs scheme, and hence for symmetric schemes with $w_1 = w_2$ sufficiently small, a positive definite form (9.10) can be achieved with a choice of weights satisfying (9.9). Thus, for class N schemes close to the Lax-Friedrichs schemes one can work with functionals such that the maximal weights associated with numerical self-interactions of shocks are less than half the weight associated with transverse group interactions.

10. STABILITY OF HYBRIDIZED SCHEMES.

In this section we shall restrict our attention to systems of two equations with eigenvalues satisfying (1.2) and $\lambda_1(u) = -\lambda_2(u)$. Our results extend to the more general class with eigenvalues satisfying (1.2) and $\lambda_1 < 0 < \lambda_2$ but we shall treat just the symmetric case $\lambda_1 = -\lambda_2$ for concreteness. We shall establish stability in the total variation norm for hybridized schemes which are based on the tracking of major waves defined by parameters k,m and $\sigma(\Delta x)$, cf. Section 8, and which employ the modified random choice generator, cf. Section 7 together with a generator of the form $\phi + q$ where ϕ corresponds to

a symmetric scheme in class N, cf. Section 9, with w_1 and w_2 sufficiently small and where q is the associated perturbation constructed in Section 6.

Theorem 10.1. If $\sigma(\Delta x)$ satisfies (7.6) then for appropriate choices of m and k, $m > k$, the hybridized schemes above produce difference approximations $u(x,t) = u(x,t;\Delta x,\Delta t)$ with the following property. If the initial data u_0 lies in a small neighborhood of a state $\bar{u} \in R^n$ and if $TVu_0 << 1$ then

$$TVu(\ ,t) \leq \text{const.}TVu_0 \tag{10.2}$$

$$\int_{-\infty}^{\infty} |u(\ ,t_1) - u(\ ,t_2)|dx \leq \text{const.}(|t_1 - t_2| + \Delta x)\ , \tag{10.3}$$

provided that t, t_1 and t_2 are less than $\text{const.}\Delta x/\sigma^2$. Furthermore, the constants depend only on $\bar{u}$, the equations and the parameters which define the scheme.

Remarks. The approximate L^1 *Lipschitz continuity* (10.3) *for* the marching map is an immediate consequence of (10.1); the difference approximation $u = u(x,t;\Delta x,\Delta t)$ is constant on each rhombus centered at a mesh point $(i\Delta x, j\Delta t)$ with sides of length $\Delta s = (\Delta x^2 + \Delta t^2)^{1/2}$ oriented by the normals α and β and assumes therein the value of the corresponding grid function at $(i\Delta x, j\Delta t)$. One may, of course, derive piecewise constant or piecewise smooth difference approximations from a given grid function in any of several standard ways and still maintain estimates of the form (10.2) and (10.3). We shall comment further on this point in Section 11. Lastly, we note that the particular choice $\sigma = \Delta x^{1-\varepsilon}$ leads to uniform estimate (10.2) and (10.3) over time intervals of length $1/\Delta x^{1-2\varepsilon}$; the ratio of meshlength is held fixed and satisfies the C-F-L condition.

Before proving Theorem (10.1) several preliminary *remarks are* in order concerning the local action of the hybridized functionals. For notational

convenience we shall work with a functional of the form

$$F(X) = cTVX + P + T$$

where c is an appropriate small constant and P and T denote potentials with weights hybridized according to the rules of Section 9. In order to distinguish various local changes in F according to the structure of the incoming waves we shall employ the notation

$$\tilde{\delta}_j = \delta_j \text{ if } \Omega^- \in RC;\ \tilde{\delta}_j = 0 \text{ if } \Omega^+ \in C$$
$$\tilde{\gamma}_j = \gamma_j \text{ if } \Omega^+ \in RC;\ \tilde{\gamma}_j = 0 \text{ if } \Omega^+ \in C$$
$$\bar{\delta}_j = \delta_j - \tilde{\delta}_j;\ \bar{\gamma}_j = \gamma_j - \tilde{\gamma}_j ,$$

if δ_j and γ_j are incoming j-waves of a mesh diamond Ω whose neighbors with a common boundary along the ws and se sides are denote by Ω^- and Ω^+ respectively. Given two consecutive I-curves $J_2 > J_1$ separated by a mesh diamond Ω with incoming waves $\nu = (\delta,\gamma)$ and local base state u_s, let $P(\nu)$ denote the incoming potential for self-interactions and $G(\nu)$ the incoming group potential:

$$P(\nu) = \sum_{j=1}^{2} \phi_j(\delta_j,u_s)\delta_j^2 + \phi_j(\gamma_j,u_s)\gamma_j^2$$

$$G(\nu) = p(u_s)|\delta_1\delta_2| + r(u_s)|\gamma_1\gamma_2| .$$

We note that $r = p$ since we are dealing with symmetric schemes. Now, if $TVJ_1 << 1$, we have the following estimates with an appropriate small constant ε:

$$F(J_1) - F(J_2) \leq -(1 - \varepsilon)D(\nu) - \varepsilon C(\nu) - P(\bar{\nu}) - G(\bar{\nu}) \tag{10.4}$$

$$F(J_1) - F(J_2) \leq -\varepsilon\tau(\nu) - \varepsilon d^2(\nu) + P(\tilde{\nu}) + G(\tilde{\nu}) + O(|\nu|^3) \tag{10.5}$$

if Ω lies respectively in RC for (10.2) and in C for (10.3) where

$$D(\nu) = s|\delta_2\gamma_1| + \sum \{|\delta_j\gamma_j| : \delta_j \text{ and } \gamma_j \text{ approach}\} .$$

The estimates (10.4) and (10.5) can be summarized by introducing a quantity

$I(\Omega)$, which equals the negative of the right hand side of (10.4) if $\Omega \in RC$ and the negative of the sum of the first four terms on the right hand side of (10.5) if $\Omega \in C$, and obtaining

$$F(J_2) - F(J_1) \leq I(\Omega) + \text{const.}\sigma^2(\Delta x)|\nu|$$

by appealing to the fact that the waves in conservative diamonds have an order at most σ. Thus, the problem of proving that F has at most mild growth along orbits requires an estimate on the changes in F due to the switches between generating functions. If Ω, Ω^- and Ω^+ all beong toeither C or if $\Omega \in RC$ then $I(\Omega) \leq 0$. Hence the only unfavorable contribution arises if Ω lies in the set K of mesh diamonds in C with at least neighbor Ω^-, Ω^+ in RC. If $\Omega \in K$ then F is augmented by the effect of switching on the coefficients associated with group and self-interactions. We shall show that for each diamond Ω in K one can associate a set of diamonds at lower time levels at which compensating interactions have occurred. Specifically, we shall show that if TVM^kX is sufficiently small for $k \leq p$ then

$$\sum \{I(\Omega) : \Omega \quad [0, k\Delta t]\} \leq 0 , \tag{10.7}$$

if $k \leq p$. Combining (10.5) with the property that

$$F(MX) \leq F(X)\{1 + \text{const.}\sigma^2(\Delta x)\} - \sum_j I(\Omega_j)$$

where $\{\Omega_j\}$ denotes the set of all mesh diamonds separating the I-curves associated with X and MX, we obtain

$$F(M^kX) \leq 2F(X) \quad \text{if} \quad p \leq \text{const.}/\sigma^2(\Delta x)$$

and $TVX \ll 1$. Furthermore, in the course of the proof of (10.7) we shall establish a slightly stronger estimate which facilitates the proof that the difference approximations converge to an exact solution. The stronger estimate assumes the form

(10.8) $$\sum I(\Omega) \leq -\varepsilon \sum I^*(\Omega)$$

where $0 < \varepsilon << 1$, both summations are taken over the mesh diamonds Ω which lie in a given strip $[0,p\Delta t]$ and the interaction term I^* is defined by

(10.9) $$I^*(\Omega) = \tau(\nu) + d^2(\nu)$$

(10.10) $$I^*(\Omega) = D(\nu) + C(\nu) + G(\nu) + P_0(\nu) + H(\nu)$$

if Ω lies respectively in C for (10.9) and RC for (10.10). Here P_0 records the self-interactions associated with those incoming waves $\nu = (\delta,\gamma)$ of Ω which have maximal coefficients and enter from a diamond in C, i.e.

$$P_0(\nu) = \bar{\delta}_1^2 + \bar{\gamma}_2^2 ,$$

and $H(\nu)$ records the effect of switching off coefficients at the starting points of major waves:

$$H(\nu) = m\sigma^2$$

if Ω contains an initial segment of a major wave while $H=0$, otherwise. The factor of m is introduced merely as a reminder of the definition of H. We deduce from (10.8) that

$$F(M^k X) \leq 2F(X) - \text{const.} \sum I^*(\Omega) \quad \text{if} \quad k \leq \text{const.}/\sigma^2(\Delta x)$$

and therefore that total amount of interaction as measured by I^* in a strip of the form $[0,p\Delta t]$ satisfies

(10.11) $$\sum \{I^*(\Omega) : \Omega \in [0,p\Delta t]\} \leq \text{const.} TVu_0$$

if $p \leq \text{const.}/\sigma^2$.

Proof of Theorem. Consider a fixed strip $[0,p\Delta t]$ and denote by K the set of mesh diamonds Ω therein such that either Ω^- or Ω^+ lies in RC. We shall indicate how to associate with each Ω in K a collection of diamonds $\tilde{\Gamma}_j = \tilde{\Gamma}_j(\Omega)$, $j = 1,2,\ldots,\ell$, and compensating quantities $q(\tilde{\Gamma}_j,\Omega)$ such that

$$
\begin{aligned}
&|q(\tilde{\Gamma},\Omega)| \leq |I(\tilde{\Gamma}_j)| \\
&I(\Omega) + \sum_j q(\tilde{\Gamma}_j,\Omega) \leq 0 \\
&\sum_{K^c} I(\Omega) \leq \sum_K \sum_j q(\tilde{\Gamma}_j,\Omega) \ .
\end{aligned} \tag{10.12}
$$

It then follows that

$$
\sum_K I(\Omega) + \sum_{K^c} I(\Omega) \leq \sum_K I(\Omega) + \sum_K \sum_j q(\tilde{\Gamma}_j,\Omega) \leq 0 \ .
$$

Here diamonds of the form $\tilde{\Gamma}_j(\Omega)$ and $\tilde{\Gamma}_k(\Omega')$ may coincide for distinct Ω and Ω'. The estimate (10.6) simply follows from the fact that compensating quantities q can be located without appealing to a small fraction of $I^*(\Omega)$.

Fix Ω in K. In the case study below we shall locate diamonds Γ_j having interactions which compensate for the contribution to $I(\Omega)$ from waving leaving Γ^+. The set of all diamonds of the form Γ_j or Γ'_j provides the desired collection $\{\tilde{\Gamma}_j\}$. Thus, let us suppose that Ω^- lies in RC and let δ_1 and δ_2 be the waves in the Riemann problem with data (u_w,u_e) where w and e denote the values of the difference approximation at the west and east vertices of Ω^-. We shall first treat the case of two shock waves. The remaining cases are somewhat easier and will be discussed below.

Case 1. $\delta_1 < 0, \ \delta_2 > 0$

Subcase 1. Suppose both δ_1 and δ_2 cross the ne side of Ω^-. If Ω^- contains an incoming major j-rarefaction wave then

$$
C(\Omega^-) + D(\Omega^-) \geq c\sigma
$$

since there exists no outgoing rarefaction wave for Ω^-. As before we shall denote by c any of various constants depending only on the scheme, the equations and the neighbor in which all analysis takes place. On the other hand

$$I(\Omega) \leq ck^2\sigma^2 .$$

Hence, in this situation we associate with Ω just a single diamond $\Gamma_1 = \Omega^-$ and put

$$q(\Gamma_1,\Omega) = \sigma C(\Omega^-) + \sigma D(\Omega^-) .$$

Now, if Ω^- contains an incoming major j-shock then

$$\sum C(\Omega_i) + D(\Omega_i) \geq c\sigma$$

where $\{\Omega_i\}$ denotes the set of diamonds which it intersects. Here, we put $\Gamma_i(\Omega) = \Omega_i$ and

$$q(\Gamma_i,\Omega) = \sigma C(\Omega_i) + \sigma D(\Omega_i) .$$

Subcase 2. Suppose δ_1 crosses the wn side of Ω^- while δ_2 crosses the ne side of Ω^-. If Ω^- contains a major 2-wave the analysis of subcase 1 is applicable. Let us therefore consider the situation where Ω^- contains a major 1-wave. Here the only non-negative contribution to $I(\Omega)$ associated with waves leaving Ω^- is given by $b_2\delta_2^2$ and appropriate compensation can be obtained as follows. Let δ be an outgoing j-shock of an arbitrary diamond in RC. We introduce a backward shock tree $T(\delta)$ through δ by starting with δ and repeating the following process: given an outgoing j-shock of a diamond in RC associate with it the corresponding set of incoming j-shocks, if any. Such a tree $T(\delta)$ is contained in a union of RC diamonds, say Ω_k, each having one outgoing j-shock and at most two incoming j-shocks. With reference to the particular shock δ_2 we shall refer to a corresponding diamond Ω_k as a terminal diamond for $T(\delta_2)$ if any of the following conditions hold:

(10.13) Ω_k^+ lies in C and contains an outgoing 2-shock, say α_2, crossing its wn side

(10.14) Ω_k^+ lies in C and contains an outgoing 2-shock, say β_2 crossing its ne side

(10.15) Ω_k contains no incoming 2-shock.

We shall refer to such associated waves α_2 and β_2 as terminal shocks for $T(\delta_2)$. We can estimate δ_2^2 in terms of terminal shocks of $T(\delta_2)$ as follows:

$$\delta_2^2 \leq \sum \alpha_2^2 + \sum \beta_2^2 + 2 \sum_k S_2(\Omega_k) + c \operatorname{osc} T(\delta_2) \sum_k D(\Omega_k) . \tag{10.16}$$

Here the first two summations are taken over all terminal shocks of $T(\delta_2)$ as indicated in(10.13) and (10.14); $S_2(\Omega_k)$ records the shock-shock interaction of the second field in Ω_k, i.e.

$$S_2(\Omega_k) = |\varepsilon\eta|$$

if Ω_k contains incoming 2-shocks ε and η while $S_2 = 0$ otherwise; and $\operatorname{osc} T(\delta_2)$ denotes the largest wave magnitude in the union of Ω_k. Thus, in the case of difference approximations with small oscillation, the sum of squares of terminal shocks plus the total shock-shock interaction on $T(\delta_2)$ bounds the "output" δ_2^2 modulo a small fraction of the total random choice interaction occurring on $T(\delta_2)$. The estimate (10.16) can be established by proving by induction that

$$\begin{aligned}\delta_2^2 \leq \sum \alpha_2^2 + \sum \beta_2^2 + 2 \sum S_2(\Omega_k) \\ + 4c \operatorname{osc} T(\delta_2) \sum D(\Omega_k) + c^2 \sum D^2(\Omega_k)\end{aligned}$$

where the constant c is chosen so large that the incoming and outgoing j-waves of a general random choice interaction satisfy

$$|\varepsilon_j - (\delta_j + \gamma_j)| \leq cD(\delta,\gamma) ,$$

cf. Section 7. At this point we remark that we can choose the weights b_2 (and a_1) to be constant since we only need (5.4) to hold. If b_2 is constant, we obtain an estimate of the form

$$\begin{aligned}b_2\delta_2^2 \leq \sum b_2\alpha_2^2 + \sum b_2\beta_2^2 + cb_2 \operatorname{osc} T(\delta_2) \sum D(\Omega_k) \\ + 2b_2 \sum S_2(\Omega_k) ,\end{aligned} \tag{10.17}$$

for which there exist obvious compensating quantities $q(\Gamma_j,\Omega)$ for the first three term on the right hand side. It remains only to obtain compensation for the last term of the right hand side of (10.17). Now, since $b_2 > 1/2$, the shock-shock interaction (with coefficient equal to one) in $D(\Omega_k)$ does not cover this last term. However, we can obtain adequate compensation for it by appealing to diamonds of the form Ω_k^-. Specifically, in subcase 2 we shall associate with a collection of diamonds Γ_j such that

$$\cup \Gamma_j \subset \cup \Omega_k \cup \Omega_k^- .$$

We partition the analysis of subcase 2 as follows.

Subcase 2A. Suppose Ω_k satisfies (10.13). Let β_1,β_2 denote the outgoing waves of Ω_k^- crossing the ws side of Ω_k and α_1,α_2 the outgoing waves of Ω_k^+ crossing the se side of Ω_k. Now if Ω_k contains a major 2-wave then we may proceed exactly as in subcase 1. Let us assume therefore that Ω_k contains a major 1-wave. If α_1 lies on a major wave then $|\alpha_1| \geq m\sigma$ and we can appeal to (a small fraction of) the quantity $-a_1\alpha_1^2$ which appears in $I(\Omega_k)$; we have

$$-a_1\alpha_1^2 \leq -a_1 m^2\sigma^2 ,$$

with m large. Hence, the only difficulty arises in the situation where we know only that β_1 lies on a major 1-wave and this can be handled as follows. Since β_1 and β_2 cross the same line segment, the definition of the modified random choice generator, cf. Section 7, implies that either

$$|\beta_1\beta_2| \leq \varepsilon\sigma^2 \quad \text{or} \quad |\beta_1| + |\beta_2| \geq L\sigma .$$

Suppose the former inequality holds. We write the term $a_2\alpha_2^2$ appearing in $-I(\Omega_k)$ in the form

$$a_2\alpha_2^2 = b_2\alpha_2^2 + (a_2 - b_2)\alpha_2^2 . \tag{10.18}$$

We recall that a_2 depends on the value of the difference approximation at the south vertex of Ω_k and satisfies $a_2 > b_2$. The first term on the right hand side of (10.18) appears in (10.17) and we have a residual of the form

$(a_2 - b_2)\alpha_2^2$ which we take advantage of as follows. We note that if

(10.19) $$(a_2 - b_2)\alpha_2^2 > 2b_2 S_2(\Omega_k)$$

we have located adequate compensation for the shock-shock interaction in Ω_k. Now (10.19) holds if

$$|\beta_2| < (a_2 - b_2)|\alpha_2|/2b_2$$

since $S_2(\Omega_k) = |\beta_2\alpha_2|$. Equivalently, (10.19) holds if

(10.20) $$\varepsilon\sigma < (a_2 - b_2)|\alpha_2|/2b_2$$

since β_1 lies on a major wave and $|\beta_1\beta_2| < \varepsilon\sigma^2$. Therefore no further analysis is required unless we are faced with the opposite inequality from (10.20), i.e. unless

(10.21) $$|\alpha_2| < 2b_2\varepsilon\sigma/(a_2 - b_2) .$$

However if (10.21) holds we can locate adequate compensation q in the diamond Ω_k^- as follows. Inequality (10.21) implies that

$$S_2(\Omega_k) < \{2b_2\varepsilon/(a_2 - b_2)\}\sigma\beta_2 .$$

In a lemma below we shall show that $|\beta_1\beta_2| < c\tilde{I}(\Omega_k^-)$ where

$$\tilde{I}(\Omega_k^-) = G(\Omega_k^-) + D(\Omega_k^-) + H(\Omega_k^-) .$$

We therefore obtain $\sigma|\beta_2| < c\tilde{I}(\Omega_k^-)$ and

(10.22) $$2b_2 S_2(\Omega_k) < \varepsilon c\tilde{I}(\Omega_k^-) .$$

We conclude from (10.22) that if $|\beta_1\beta_2| < \varepsilon\sigma^2$ there exists an appropriate compensating quantity q for the shock-shock interaction in Ω_k since we have an estimate of the form

(10.23) $$b_2\alpha_2^2 + 2b_2 S_2(\Omega_k) < \varepsilon c\tilde{I}(\Omega_k^-) + a_2\alpha_2^2$$

with small ε. We note that (10.13) can be strengthened by the inclusion on the left hand side of a term of the form $c\alpha_2^2$. This can easily be seen by writing (10.18) in the form

$$a_2\alpha_2^2 = b_2\alpha_2^2 + (a_2 - b_2)\alpha_2^2/2 + (a_2 - b_2)\alpha_2^2/2$$

and proceeding as above. This accounts for the presence of the term $P_0(v)$ in

(10.10). We complete the discussion of subcase 2A by considering the alternative situation where $|\beta_1| + |\beta_2| > L\sigma$. Here, if $|\beta_2| > m\sigma$ we can appeal to the cancellation process exactly as in subcase 1. On the other hand if $|\beta_2| < m\sigma$ we obtain

$$\tilde{I}(\Omega_k^-) > c(L - m)\sigma^2$$

since

$$|\beta_1\beta_2| \leq c\tilde{I}(\Omega_k^-) \quad \text{and} \quad |\beta_1| > (L - m)\sigma$$

Thus if L is sufficiently large it follows that $I(\Omega)$ is bounded by a small fraction of $\tilde{I}(\Omega_k^-)$ and the desired compensation is realized.

Next we shall establish the lemma required in subcase 2A.

Lemma. If ε_1 and ε_2 denote a pair of outgoing waves crossing either the wn or ne side of a diamond Ω in RC then

$$|\varepsilon_1\varepsilon_2| \leq c\tilde{I}(\Omega) \equiv c\{G(\Omega) + D(\Omega) + H(\Omega)\}$$

Proof: Let δ_1, δ_2 denote the incoming waves crossing the ws side of Ω and γ_1, γ_2 the incoming waves crossing the se side of Ω. We have

$$|\varepsilon_j - (\delta_j + \gamma_j)| \leq cD(\Omega) .$$

If Ω^+ and Ω^- both lie in RC the structure of the modified random choice generator implies that $\delta_1 = \gamma_2 = 0$ and hence

$$|\varepsilon_1\varepsilon_2| \leq c|\delta_2\gamma_1| = cD(\Omega) .$$

If $\Omega^- \in C$ and $\Omega^+ \in RC$ then $\gamma_2 = 0$ and we obtain

$$|\varepsilon_1\varepsilon_2| \leq |\delta_1\delta_2| + cD(\Omega) \leq cG(\Omega) + cD(\Omega) .$$

Similarly if $\Omega^+ \in C$ and $\Omega^- \in RC$ then

$$|\varepsilon_1\varepsilon_2| \leq cG(\Omega) + cD(\Omega) .$$

Finally, if Ω^+ and Ω^- both lie in C we obtain the simple estimate

$$|\varepsilon_1\varepsilon_2| \leq cm^2\sigma^2 \leq cH(\Omega) .$$

The proof of the lemma is complete.

Subcase 2B. Suppose that Ω_k satisfied (10.14), as in subcase 2A the situation if simple unless we know only that Ω_k contains a major 1-wave. Supposing this to be the case, we proceed as follows. If β_1 lies on a major 1-wave then $|\beta_1| \geq m\sigma$ and we can appeal to (a small fraction of) the quantity $-b_1\beta_1^2$ which appears in $I(\Omega_k)$ and obtain

$$-b_1\beta_1^2 \leq -b_1 m^2\sigma^2$$

for appropriate compensation. Therefore, let us assume, on the other hand, that α_1 lies on a major 1-wave and let us estimate $S_2(\Omega_k)$. Now $S_2(\Omega_k)$ vanishes if $\alpha_2 = 0$. If $\alpha_2 \neq 0$ then either

$$|\alpha_1\alpha_2| \leq \varepsilon\sigma^2 \quad \text{or} \quad |\alpha_1| + |\alpha_2| \geq L\sigma .$$

If the former inequality holds then

$$S_2(\Omega_k) = |\alpha_2\beta_2| \leq \varepsilon\sigma|\beta_2| \leq \varepsilon|\alpha_1\beta_2| \leq \varepsilon D(\Omega_k) .$$

since $|\alpha_1| \geq \sigma$. If the latter inequality holds then we remark that either $|\alpha_2| \geq m\sigma$ in which case we can appeal to the cancellation process exactly as in subcase 1 or $|\alpha_2| < m\sigma$ in which case

$$D(\Omega_k) \geq |\beta_2\alpha_1| \geq |\beta_2|(L - m)\sigma \geq (L - m)|\beta_2\alpha_2|/m$$

since $|\alpha_1| \geq (L - m)\sigma$. Hence if L is sufficiently large

$$2b_2S_2(\Omega_k) \leq \frac{1}{m} D(\Omega_k)$$

and we conclude that a small fraction of the random choice interaction in Ω_k provides adequate compensation for $S_2(\Omega_k)$ (if m is large).

Subcase 2c. Suppose Ω_k is not a terminal diamond. Then Ω_k^+ and Ω_k^- both lie in RC and we proceed as follows. If $\beta_1 \neq 0$ then $\alpha_2 = 0$ and no further analysis is necessary. If $\beta_1 = 0$ then we may assume that $|\alpha_1| > \sigma$. Now, if $\alpha_2 < 0$ then (10.24) holds. Under the first inequality of (10.24) we obtain an adequate estimate of the form

$$2b_2S_2(\Omega_k) = 2b_2|\alpha_2\beta_2| \leq \varepsilon cD(\Omega_k) ,$$

since

$$D(\Omega_k) \geq s|\beta_2\alpha_1| \geq s\sigma|\beta_2| \ .$$

Suppose the second inequality of (10.24) holds. If $|\alpha_2| \geq m\sigma$ then no further analysis is required as our previous cases apply. If $|\alpha_2| < m\sigma$ then $|\sigma_1| \geq (L - m)\sigma$ and we obtain

$$D(\Omega_k) \geq s|\beta_2\alpha_1| \geq s(L - m)\sigma|\beta_2| \geq s(L - m)|\alpha_2\beta_2|/2$$
$$= s(L - m)\{2b_2S_2(\Omega_k)\}/4b_2 \ ,$$

since we may assume that $|\alpha_2| \leq 2\sigma$, otherwise the situation if trivial. Hence, if L is sufficiently large, a small fraction of $D(\Omega_k)$ dominates the shock-shock interaction of the second field in Ω_k. This completes the proof of case 1.

We note that the presence of the terms (10.9) and (10.10) in $I^*(\Omega)$ is simply due tothe fact that we were able to obtain appropriate compensation for the positive terms in $I(\Omega)$ by appealing to only a fraction (less than one) of the available negative terms.

It remains to discuss the cases where the Riemann problem with data (u_w,u_e) produces a shock and a rarefaction wave or a pair of rarefaction waves. We note that if the strip under consideration involves no splitting of rarefaction waves by the sample point then the analysis of case 1 applies virtually without modification to the remaining cases. If a sample point splits a rarefaction wave, only a small change in the above analysis is required since the splitting of rarefaction waves is not by itself produce any wave amplification. In this connection we note that it is primarily to handle one technical point connected with wave splitting that we restrict attention to symmetric schemes with w_1 and w_2 small.

We begin with a remark needed to construct the analogous rarefaction trees. Let α_j and β_j denote the j-waves crossing the wn and ne side of

a diamond Ω in RC and δ_j and γ_j the j-waves crossing the ws and se sides respectively. Suppose that $\alpha_j > 0$, $\beta_j > 0$ and consider an ordered partitioning of α_j and β_j of the form

$$\alpha_j = \sum_{k=1}^{m} \alpha_{jk}; \quad \beta_j = \sum_{k=1}^{n} \beta_{jk}$$

where α_{jk} and β_{jk} are j-rarefaction waves contained in α_j and β_j respectively such that the left edge of α_{jk} coincides with the right edge of $\alpha_{j,k+1}$ while the left edge of β_{jk} coincides with the right edge of $\beta_{j,k+1}$. It is easy to see that there exist partitions of the incoming j-rarefaction waves (if any) which conserve wave magnitudes modulo a quantity on the order of $D(\Omega)$ as follows. Let

$$\pi_{jk} = \alpha_{jk} \text{ if } 1 \leq k \leq m; \quad \pi_{jk} = \beta_{j,k-m} \text{ if } m+1 \leq k \leq n$$

Case A. If $\delta_j > 0$, $\gamma_j > 0$ then there exists a j-rarefaction wave δ'_j contained in δ_j which can be partition into j-rarefaction waves δ'_{jk}, $1 \leq k \leq p$, in such a way that

$$\delta'_j = \sum_{k=1}^{p} \delta'_{jk} \quad \text{and} \quad \sum_{k=1}^{q} |\pi_{jk} - \delta'_{jk}| + \sum_{q+1}^{p} |\pi_{jk}| \leq cD(\Omega) .$$

for appropriate q. It is understood that one may have $q = p$ in which case the third summation in (10.25) is taken to vanish.

Case B. If $\delta_j < 0$, $\gamma_j < 0$ then there exists a j-rarefaction γ'_j in γ_j which can be partitioned into γ'_{jk} so that

$$\gamma'_k = \sum_{k=1}^{p} \gamma'_{jk} \quad \text{and} \quad \sum_{k=1}^{q} |\pi_{jk} - \gamma'_{jk}| + \sum_{q+1}^{p} |\pi_{jk}| \leq cD(\Omega) .$$

Case C. If $\delta_j > 0$, $\gamma_k > 0$ then partitions

$$\delta_j = \sum_{k=1}^{p} \delta_{jk} \quad \text{and} \quad \gamma_j = \sum_{j=1}^{q} \gamma_{jk}$$

can be formed in such a way that either

$$(10.26) \qquad \sum_{k=1}^{p} |\pi_{jk} - \delta_{jk}| + \sum_{p+1}^{q} |\pi_{jk} - \gamma_{j,k-p}| \leq cD(\Omega), \quad \text{or}$$

$$(10.27) \qquad \sum_{k=1}^{p-1} |\pi_{jk} - \delta_{jk}| + \sum_{p+1}^{q} |\pi_{jk} - \gamma_{j,k-p+1}| + |\pi_{jp} - \delta_{jp} - \gamma_{j1}| \leq cD(\Omega) .$$

We note that virtually the same magnitude-preserving partitions are employed in [13] for the purpose of constructing characteristic curves. Using such partitions, an analogous backward rarefaction tree through a given outgoing rarefaction wave of a diamond Ω in RC can be constructed by repeating the following process which associates with a given outgoing j-rarefaction wave π_{jk} the corresponding incoming wave or waves according to the following rules

In Case A: $\pi_{jk} \sim \delta'_{jk}$

In Case B: $\pi_{jk} \sim \gamma'_{jk}$

In Case C: If (10.26) holds then

$$\pi_{jk} \sim \delta_{jk} \quad \text{if} \quad k \leq p \quad \text{and} \quad \pi_{jk} \sim \gamma_{j,k-p} \quad \text{if} \quad k \geq p + 1 .$$

If (10.27) holds then

$$\pi_{jk} \sim \delta_{jk} \quad \text{if} \quad k \leq p - 1 \quad \text{and} \quad \pi_{jk} \sim \gamma_{j,k-p+1} \quad \text{if} \quad k \geq p + 1$$

$$\pi_{jk} \sim (\delta_{jp}, \gamma_{j1}) .$$

Thus, only in Case C, (10.27) does not associate a pair of waves with a given outgoing wave. We now proceed to sketch the remaining cases.

Case 2. $\delta_1 > 0$, $\delta_2 < 0$.

Subcase 1. Suppose that δ_1 crosses the wn side of Ω^- while δ_2 crosses the ne side of Ω^-. This subcase can be handled exactly as in Case 1.

Subcase 2. Suppose that δ_1 is split by the sample point into a 1-wave δ_1' crossing the wn side of Ω^- and a 1-wave δ_1'' crossing the ne side of Ω^- along with δ_2: $\delta_1 = \delta_1' + \delta_1''$. Here we must locate compensation for terms of the form

$$b_1\delta_1''^2 + p\delta_1''\delta_2$$

which enter $I(\Omega)$. Now if Ω^- contains either an incoming major 2-wave or an incoming major 1-shock the analysis of Case 1 suffices. Let us therefore assume that Ω^- contains an incoming major 1-rarefaction wave. Let β_1,β_2 denote the incoming waves of Ω^- crossing its ws side and α_1,α_2 the incoming waves of Ω crossing its se side. Let Ω_1 and Ω_2 denote the diamonds which intersect the ws and se sides of Ω^- respectively. For concreteness we treat the situation where Ω_1 and Ω_2 both lie in RC; this case provides the primary example of the switching-on of weights due to the splitting of a rarefaction wave. Here we proceed by tracing back the major 2-rarefaction wave. Here we proceed by tracing back the major 2-rarefaction wave through α_1 to the diamond Γ which contains its initial segment ε. Now, if ε enters Γ from a diamond in C then $\varepsilon \geq m\sigma$ and we can appeal to a quantity on the order of ε^2 appearing in $I(\Gamma)$. On the other hand, if ε enters Γ from a diamond in RC then $\varepsilon > k\sigma$ and we shall show that there again exists compensating interactions on the order of ε^2 in certain diamonds near Γ to be described below.

Before proceeding with the latter case we shall remark on the situation where more than one major rarefaction wave is traced back to the same diamond Γ. Now, if $\{\varepsilon_j\}$ denotes the collection of all waves of the above type which

can be traced back to the same ε, then, modulo a quantity equal to a small fraction of the random choice interaction in the associated diamonds, we have

$$\varepsilon^2 \geq m \sum \varepsilon_j^2 \quad \text{if} \quad \varepsilon \geq m\sigma \quad \text{and} \quad \varepsilon^2 \geq k \sum \varepsilon_j^2 \quad \text{if} \quad \varepsilon \geq k\sigma ,$$

since in both situations $\varepsilon_j \leq \sigma$. Indeed, without loss of generality we have

$$\varepsilon = \sum \varepsilon_j$$

modulo interactions, and therefore in the case, say, $\varepsilon \geq m\sigma$,

$$\varepsilon^2 = \sum \varepsilon_j^2 + \varepsilon \sum \varepsilon_j(1 - \varepsilon_j/\varepsilon) \geq \sum \varepsilon_j^2 + (1 - 1/m)\varepsilon^2 .$$

Thus, the splitting of rarefaction waves produces a situation where the sum of the squares of the terminal waves ε_j associated with the switching-on of weights is a small fraction, here $1/m$ or $1/k$ with m and k large, of the square of the initial wave ε, modulo a quantity on the order of the same small fraction of the total random choice interaction as measured by D in the union of diamonds through which the traced waves pass.

Our analysis of the case where ε enters Γ from a diamond in C is subdivided as follows.

Subcase 2A. Suppose Γ^- and Γ^+ both lie in RC let (δ_1,δ_2) and (γ_1,γ_2) denote the incoming waves of Γ crossing its ws and se sides. We have

$$\varepsilon = \delta_1 + \gamma_1 + OD(\Gamma) . \tag{10.28}$$

Now if $\delta_1 = \gamma_1 = 0$ then $|\varepsilon| \leq cD(\Gamma)$ and we easily obtain appropriate compensation of the form ε^2 by appealing to a small fraction of $D(\Gamma)$. Indeed we have

$$\varepsilon^2 \leq c(\text{osc } u)D(\Gamma) ,$$

since $D(\Gamma)$ is itself on the order of the oscillation of the difference approximation u. Let us therefore suppose that δ_1 and γ_1 do not both vanish. For concreteness assume $\delta_1 \neq 0$. It then follows from the definition of the modified random choice generator that $\gamma_2 = 0$. If $\gamma_1 = 0$ then

$\varepsilon_1 = \delta_1$ contradicting the fact that ε is the initial segment of a major rarefaction wave. On the other hand if $\gamma_1 \neq 0$ then a simple calculation shows that

$$|\delta_1| + |\gamma_1| \leq c|\delta_2| \,, \tag{10.29}$$

since both δ_1 and γ_1 are non-zero rarefaction waves. In this connection we note if $\delta_2 = 0$ then at least one of the waves δ_1, γ_1 must vanish since the associated eigenvalue λ_1 increases from left to right across 1-rarefaction waves. It follows from (10.28) that

$$D(\Gamma) + \tilde{I}(\Gamma^-) \geq c|\delta_2\gamma_1| + c|\delta_1\delta_2| \geq c(|\delta_1| + |\gamma_1|)^2$$

and we conclude that

$$\varepsilon^2 \leq c\{D(\Gamma) + \tilde{I}(\Gamma^-)\} \,,$$

to which we appeal for appropriate compensation. The situation where $\gamma_1 \neq 0$ is handled in a similar fashion.

Subcase 2B. Suppose Γ^- and Γ^+ both lie in C. If $|\delta_1| \leq |\gamma_1|$ we appeal to $-a_1\gamma_1^2$ appearing in $I(\Gamma)$ and obtain

$$\varepsilon^2 \leq 8\gamma_1^2 + 2cD(\Gamma^-)$$

using (10.28). Hence a quantity on the order of

$$-a_1\gamma_1^2 + (\text{osc } u)D(\Gamma)$$

bounds ε^2. If $|\delta_1| \geq |\gamma_1|$ we proceed as follows. Either γ_2 or δ_2 lies on a major 2-wave. If γ_2 does then

$$\varepsilon \leq 2k\sigma + cD(\Gamma) \tag{10.30}$$

since $\delta_1 \leq k\sigma$. In this situation we appeal to the quantity $-a_2\gamma_2^2$ appearing in $I(\Gamma)$ and note that modulo a small fraction of $D(\Gamma)$ we have an estimate of the form

$$a_2\gamma_2^2 \geq a_2m^2s^2 \geq (a_2m^2/4k^2)\varepsilon^2$$

using $|\gamma_2| \geq m\sigma$. Here we are led to take m much larger than k. Next,

suppose that δ_2 lies on a major 2-wave. In this situation we simply repeat the argument given in subcases 2A and 2B and in subcase 2C below with Γ^- replacing Γ and use the fact that, modulo the interaction term $D(\Gamma)$ we have $\delta_1 \geq k\sigma/2$ since $|\delta_1| \leq |\gamma_1|$.

Subcase 2C. Suppose Γ^- lies in C while Γ^+ lies in RC. If $|\delta_1| \geq |\gamma_1|$ we appeal to $-b_1\delta_1^2$ appearing in $I(\Gamma)$ and obtain an inequality of the form

$$b_1\delta_1^2 \geq b_1\varepsilon^2/4 - cD^2(\Gamma) .$$

Next, let us suppose that $|\delta_1| < |\gamma_1|$. If δ_2 lies on a major wave then we appeal to $-b_2\delta_2^2$ appearing in $I(\Gamma)$ as follows. We have (10.30) since $\delta_1 \leq k\sigma$ and $|\delta_1| < |\gamma_1|$. Therefore

$$b_2\delta_2^2 \geq b_2m^2\sigma^2 \geq b_2m^2\varepsilon^2/4k^2 - cD^2(\Gamma) .$$

On the other hand if γ_2 lies on a major 2-wave we simply repeat the arguments above replacing Γ with Γ^- and using the fact that

$$\gamma_1 \geq k\sigma/2 - cD(\Gamma)$$

This completes the sketch for the three subcases.

The remaining term $p\delta_1''\delta_2$ can be handled as follows. From the definition of the modified random choice generator we have $\alpha_1 = \beta_2 = 0$ since Ω_1 and Ω_2 lie in RC. Thus, a term of the form

$$s|\alpha_2\beta_1| + ck\delta_1''^2$$

is available to compensate for $p\delta_1''\delta_2$. Writing $\delta_1'' = \tau\beta_1$ with $0 \leq \tau \leq 1$, we require an inequality of the form

(10.31) $$p|\delta_1''\delta_2| = p\tau|\alpha_2\beta_1| \leq s|\beta_1\alpha_2| + ck\tau^2\beta_1^2 .$$

But since $|\beta_1/\alpha_2| \geq 1/k$, (10.31) is equivalent to the condition

(10.32) $$p\tau \leq s + c\tau^2$$

Now an appropriate restriction on the C-F-L number guarantees that p is only slightly larger than s which implies that (10.32) is valid. We lastly remark

that the case where $\delta_1 < 0$ and $\delta_2 > 0$ and the case where $\delta_1 > 0$ and $\delta_2 > 0$ are treated in the same fashion as above.

11. CONVERGENCE OF HYBRIDIZED APPROXIMATIONS.

In this section we shall consider the class of systems and hybridized schemes for which we established stability in Section 10 and show, under the additional hypothesis that the error function ψ associated with the equidistributed sequence and the switching function $\sigma(\Delta x)$ satisfy

$$\lim_{n\to\infty} \psi(n)n^{3/4} = 0 \quad \text{and} \quad \lim_{\Delta x\to 0} \psi(1/\Delta x)/(\Delta x\sigma)^{1/2} = 0 ,$$

that a subsequence of difference approximations converges to an exact solution. We note that standard arguments using the stability estimates (10.2) and (10.3) yield the existence of a subsequence of difference approximations converging pointwise a.e. to a function u which is a function of bounded variation in the sense of Tonelli-Cesari [10, 33]; indeed, the subsequence converges to u in L^1 of the space variable for each fixed time t. We also note in passing that our stability analysis shows that the total variation of the difference approximations along (space-like) lines with speed of propagation greater than or equal to $\Delta x/\Delta t$ in absolute value is uniformly bounded and consequently that there exists approximate L^1 Lipschitz continuity in directions normal to such lines. It remains to prove that u satisfies (1.1) in the sense of distributions, equivalently that the contour integral

$$\int_C v(u)ds; \quad v(u) \equiv v_t u + v_x f(u)$$

vanishes for all piecewise smooth closed contours C. As we remarked earlier,

one need only verify this property for a substantial class of contours, for example, parallelograms with sides parallel to two fixed directions.

We shall begin with several comments which will facilitate the convergence proof.

Conservative Schemes. Consider a conservative scheme based on a centered diamond shaped stencil with a generating function ϕ derived from an equation of the form

$$H(u_n,u_e) - H(u_w,u_s) + G(u_n,u_w) - G(u_e,u_s) = 0 ,$$

where H and G are defined by (2.5) using normals α and β. Let $\Omega(i,j)$ denote the mesh diamond centered at $(i\Delta x, j\Delta t)$ and regard it as the union of four congruent triangular regions T_{ne}, T_{nw}, T_{ws} abd T_{es} obtained by intersection with the four standard quadrants centered at $(i\Delta x, j\Delta t)$; here the triangles share a common vertex and have hypothenuses indicated by the subscripts. Suppose that the values of all difference approximatins under consideration lie in a small neighborhood N of a fixed state $\bar{u} \in R^n$. A simple application of the implicit function theorems shows that if the matrices

$$\alpha_t I + \alpha_x \nabla f(\bar{u}) \quad \text{and} \quad \beta_t I + \beta_x \nabla f(\bar{u})$$

are invertible, i.e. if the C-F-L condition holds, then there exist smooth maps θ and ψ defined in a neighborhood of $(\bar{u},\bar{u})$ such that

$$H(a,b) = \alpha_t \theta(a,b) + \alpha_x f\{\theta(a,b)\}, \ \eta(a,a) = a$$
$$G(a,b) = \beta_t \psi(a,b) + \beta_x f\{\psi(a,b)\}, \ \psi(a,a) = a .$$

Using θ and ψ one can associate with a grid approximation $u(\Delta x)$ of the scheme, an everywhere defined piecewise constant function $u = u(x,t,\Delta x)$ such that

$$\int_C \nu(u) ds = 0$$

if C is any closed polygonal arc consisting of line segments joining adjacent mesh points: if we put $u = \theta(u_n,u_e)$ in T_{ne}, $u = \psi(u_n,u_w)$ in T_{nw}, $u = \theta(u_w,u_s)$ in T_{ws} and $u = \psi(u_e,u_s)$ in T_{es} we obtain

$$\int_{\partial\Omega(i,j)} \nu(u)ds = 0 ,$$

for all mesh diamonds Ω and consequently (11.2) by addition. Now, if $u(\Delta x)$ satisfies uniform estimates of the form (10.2) and (10.3) then so does $u(x,t,\Delta x)$ and it follows that a subsequence converges to function say $\tilde{u}$ such that

$$\int_C \nu(\tilde{u})ds = 0 \tag{11.3}$$

for all parallelograms C in $t > 0$ with normals α and β. Thus, $\tilde{u}$ is an exact solution.

Perturbations of Conservative Schemes. A similar argument applies to the perturbed schemes introduced in Section 6 with generating functions of the form $\phi + q$ where

$$|q(u_w,u_s,u_e)| \leq \text{const.}(|u_w - u_s|^2 + |u_s - u_e|^2) .$$

Using the function η and ψ associated as above with the generator ϕ we obtain

$$\int_{\partial\Omega(i,j)} \nu(u)ds = O(\Delta x)(|u_w - u_s|^2 + |u_s - u_e|^2) \tag{11.4}$$

for the corresponding piecewise constant approximation $u = u(x,t,\Delta x)$ obtained by setting $u = \theta(u_n,u_e)$ in T_{ne}, etc. Therefore, if $u(\Delta x)$ satisfies uniform estimates of the form (10.2) and (10.3) it follows that a subsequence of the functions $u(x,t,\Delta x)$ converges to a function $\tilde{u}$ satisfying (11.3) for all associated parallelograms C provided that the maximum magnitude of interpolated waves vanishes as the mesh is refined, i.e.

$$\lim_{\Delta x \to 0} \sup\{|u(x,t) - u(x + \Delta x, t + \Delta t)| + |u(x,t) - u(x + \Delta x, t - \Delta t)|\} = 0$$

where the supremum is taken over all mesh points (x,t), say in a given bounded set. Thus, a stable and convergent sequence of difference approximations for a perturbed scheme yields an exact solution provided that the maximum magnitude of waves in any bounded set of $t \geq 0$ vanishes as the mesh is refined.

The Random Choice Scheme. With a grid function $u(\Delta x)$ produced by the random choice method we shall associate the standard piecewise, smooth approximtion $u(x,t,\Delta x)$ as follows. Let

$$R(a,b) = \{(x,t) : a - \Delta x < x < a + \Delta x,\ b < t < b + \Delta t\} .$$

Let $u_1 = u_1(x,t)$ denote the restriction to

(11.5) $\qquad R(i\Delta x, j\Delta t) \quad \Omega(i,j)$

of the solution to the Riemann problem at time $t = j\Delta t$ z with data (u_w,u_n) having a jump at $x = i\Delta x$, where u_w and u_n denote the values of $u(\Delta x)$ at the west and north vertices of $\Omega(i,j)$. Similarly, let $u_2 = u_2(x,t)$ denote the restriction to

(11.6) $\qquad R\{(i - 1)\Delta x, (j - 1)\Delta t\} \quad \Omega(i,j)$

of the solution to the Riemann problem at time $t = (j - 1)\Delta t$ with data (u_w,u_s) jumping at $x = (i - 1)\Delta x$; u_3 the restriction to

(11.7) $\qquad R\{(i + 1)\Delta x, (j - 1)\Delta t\} \quad \Omega(i,j)$

of the solution of the Riemann problem with data (u_s,u_e) at $t = (j - 1)\Delta t$, $x = (i + 1)\Delta x$. Define $u(x,t,\Delta x)$ in $\Omega(i,j)$ to be u_1, u_2 or u_3 according to the location of (x,t) in one of the three corresponding subregions (11.5), (11.6) and (11.7) (we will shortly employ the definition above in the setting of hybridized schemes, for those diamonds $\Omega(i,j)$ in RC).

Now the integral around the boundary of a typical diamond $\Omega(i,j)$ can be expressed in terms of the location of the corresponding sample point as follows:

(11.8) $$\int_{\partial\Omega(i,j)} v(u)ds = E^+(i-1,j-1) + E^-(i+1,j-1)$$

where

$$E^+ = \int_0^{\Delta x} u_n - R(x/\Delta t; u_w, u_e)dx, \quad E^- = \int_{-\Delta x}^{0} u_n - R(x/\Delta t; u_w, u_e)dx .$$

Here u_w and u_e denote the values of the corresponding and function at the west and east vertices of $\Omega(i,j)$. Now if Ω is parallelogram in the $x - t$ plane which can be represented as a union of mesh diamonds $\Omega(i,j,\Delta x)$ for each Δx, i.e.

(11.9) $$\Omega = \cup\{\Omega(i,j,\Delta x) : (i,j) \in A(\Delta x)\}$$

for an appropriate index set $A(\Delta x)$ then

$$\sum_{(i,j)\in A(\Delta x)} E^+(i-1,j-1) + E^-(i+1,j-1)$$

approaches zero as the mesh is refined by results on Liu [26], if $TVu_0 << 1$. It follows that the limit of sequence of random choice approximations converges to an exact solution $\tilde{u}$. We note that large parallelograms of the form (11.9) exist if we choose for example a sequence of mesh lenghts of the form $(\Delta x)_n = \text{const.}/2^n$. For general mesh lengths a trivial modification of the above argument is required to obtain the same conclusion.

Hybridized Schemes. Here we shall associate with a given grid function $u(\Delta x)$ a piecewise smooth approximation $u(x,t;\Delta x)$ by following the procedure above for conservative (or what is the same perturbed conservative) schemes if $\Omega(i,j)$ lies in C and the procedure for the random choice method if $\Omega(i,j)$ lies in RC. For such $u(x,t,\Delta x)$ the contour integral around the boundary of a fixed parallelogram Ω of the form (11.9) can be expressed as the sum of three terms I_1, I_2 and I_3. The first term I_1 records the contribution from diamonds $\Omega(i,j)$ in C due to the perturbation q:

$$I_1 = O[\Delta x \sum \{\epsilon^2 : \epsilon \text{ crossing a diamond } \Omega(i,j) \text{ in } C\}] .$$

The second term I_2 records the contribution along the boundary B between the two regions formed with mesh diamonds in C and RC respectively and takes the form

$$I_2 = O(\Delta x \cdot TVu)_B .$$

The third term represents the random choice error and takes the form

(11.10) $I_3 = \sum \{E^+(i-1,j-1) + E^-(i+1,j-1) : \Omega(i,j) \in RC \quad \Omega\} + m(\Delta x)$,

where $m(\Delta x)$ represents the alterations to the standard random choice error due to the use of the modified random choice generator; as we showed in Section 7, $m(\Delta x)$ vanishes as the mesh is refined.

It follows immediately from (10.2) that I_1, has the order $\sigma(\Delta x)$ and thus vanishes in the limit. We shall show below that the total variation of the difference approximations $u(x,t,\Delta x)$ around the boundary $B = B(\Delta x)$ has the order $o(1/\Delta x)$ and thus I_2 vanishes in the limit. Finally, the argument of Liu for the random choice method applies with an only trivial modification to show that I_3 vanishes in the limit.

The analysis of I_2 proceeds as follows. Fix a difference approximation $u = u(x,t,\Delta x)$ and a time strip $[0,T]$. Let K_1 denote the set of all mesh diamonds in C $[0,T]$ such that either Ω^- or Ω^+ lies in RC and B_1 the set of all waves ε with the following three properties: ε is incoming with respect to a diamond Ω in K_1; if ε crosses the ws side of Ω then Ω^- lies in RC; if ε crosses the se side of Ω then Ω^+ lies in RC. Let K_2 denote the set of all mesh diamonds Ω in $RC \cap [0,T]$ such that either Ω^- or Ω^+ lies in ε and let B_2 denote the set of all waves ε with the following three properties: ε is an incoming wave with respect to a diamond Ω in K_2; if ε crosses the ws side of Ω then Ω^- lies in C, if ε crosses the se side of Ω then Ω^+ lies in C. We have $B = B_1 \cup B_2$. For

simplicity we suppress the dependence of B_j on the mesh length. We shall show that

(11.10) $$\lim_{\Delta x \to 0} \Delta x TV_{B_1(\Delta x)} u(\cdot,\cdot,\Delta x) = 0 ,$$

a similar analysis produces the same result for B_2. To this end, let us fix a difference approximation $u(x,t,\Delta x)$ and a diamond Ω in B_1. We shall analyze the contribution due to waves crossing the ws side of Ω; the se side is treated in a similar fashion. Let us therefore assume that Ω^- lies in RC and for concreteness that the Riemann problem with data (u_w,u_e), where u_w and u_e denote the values of u at the west and east vertices of Ω^-, gives rise to two shock waves. Let (δ_1,δ_2) and (γ_1,γ_2) denote the incoming waves of Ω^- crossing its ws and se sides; let (α_1,α_2) and (β_1,β_2) denote the outgoing waves of Ω^- crossing its wn and ne sides. We consider the following cases.

Case 1. Suppose that a major wave terminates in Ω^-. Then the total strength S of waves leaving Ω^- and entering Ω is less than $c\sigma$. If we associate with Ω the diamond Γ which contains the initial segment of the major wave terminating in Ω^- we have $H(\Gamma) \geq c\sigma^2$ and therefore

$$\Delta x S \leq c\Delta x\sigma \leq c\Delta x H(\Gamma)/\sigma$$

Since the sum of squares of all initial segments of major waves is finite, i.e.,

$$\sum H(\Gamma) \leq c$$

it follows that the total contribution from all waves of the above type (in a fixed strip $[0,T]$) satisfies

$$\Delta x \sum S \leq c\Delta x \sum H(\Gamma)/\sigma \leq c\Delta x/\sigma$$

and therefore vanishes by (7.9).

Case 2. Suppose that no major wave terminates in Ω^-. We may assume that Ω^- contains a major 1-wave and that a 1-shock α_1 crosses the wn side of Ω^- while a 2-shock crosses the ne side of Ω^-. Let Ω_1 and Ω_2 denote the diamonds whose ne and wn sides coincide respectively with the ws and se sides of Ω^-. We shall first treat the subcase where Ω_1 and Ω_2 both lies in RC. Here we have

$$\beta_2 = \delta_2 + \gamma_2 + OD(\Omega^-) .$$

Now if γ_1 lies on a major wave then $|\gamma_1| \geq \sigma$ and we obtain an estimate of the form

$$|\sigma\beta_2| \leq |\gamma_1\delta_2| + |\gamma_1\gamma_2| + O\{\sigma D(\Omega^-)\}$$
$$\leq c\{D(\Omega^-) + \tilde{I}(\Omega_2)\} .$$

If δ_1 lies on a major wave then $|\delta_1| \geq \sigma$ and we obtain

$$|\sigma\delta_2| \leq |\delta_1\delta_2| \leq c\tilde{I}(\Omega_1)$$

But, if $\delta_1 \neq 0$ then $\gamma_2 = 0$ and we obtain

$$|\sigma\beta_2| \leq |\sigma\delta_2| + O\{\sigma D(\Omega^-)\} \leq c\{D(\Omega^-) + \tilde{I}(\Omega_2)\} .$$

Thus, the total strength of waves corresponding to the subcase where Ω_1 and Ω_2 both lie in RC is bounded by const./σ.

The subcase where Ω_1 and Ω_2 both lie in C is even simpler since we have an estimate of the form

$$|\beta_2| \leq c\sigma \leq cH(\Omega^-)/\sigma$$

using the fact that $H(\Omega^-) \geq c\sigma^2$.

If $\Omega_1 \in C$ and $\Omega_2 \in RC$ we proceed as follows. If $|\gamma_1| \geq \sigma$ we remark that

$$|\sigma\beta_2| \leq c\{D(\Omega^-) + \tilde{I}(\Omega_3)$$

as in the first subcase, while if $|\delta_1| \geq \sigma$ we have

$$|\sigma\beta_2| \leq cH(\Omega^-)$$

as in the second subcase. It remains only to treat the subcase where

and $\Omega_2 \in C$. Now if γ_1 lies on a major wave then the analysis of the first subcase applies. If δ_1 lies on a major wave then we proceed as follows. Let $\mu(\Delta x)$ be a positive function such that

$$\lim_{\Delta x \to 0} \mu(\Delta x) = 0 = \lim \Delta x/\mu\sigma$$

We distinguish two situatins accordingly as $|\beta_2| \geq \mu\sigma$ and $|\beta_2| < \mu\sigma$. In the first situation we have

$$\Delta x|\beta_2| = (\Delta x/|\beta_2|)\beta_2^2 \leq c\Delta x\{\delta_2^2 + \gamma_2^2 + D(\Omega^-)\}/\mu\sigma \ .$$

We observe that $|\delta_2| \leq c\tilde{I}(\Omega_1)/\sigma$ using the fact that

$$|\delta_1\delta_2| \leq c\tilde{I}(\Omega_1) \ .$$

But we need only treat the case where $|\delta_2| \leq c\sigma$. Therefore

$$\delta_2^2 \leq c\tilde{I}(\Omega_1)$$

and

$$\Delta x|\beta_2| \leq c\Delta x\{\tilde{I}(\Omega_1) + I^*(\Omega^-)\}/\mu\sigma$$

since γ_2 is associated with a maximal weight. In the situation where $|\beta_2| \leq \mu\sigma$ we have

$$|\beta_2| \leq \mu|\alpha_1|$$

and consequently the sum of all such waves β_2 in a given strip, say $[t,t + \Delta t]$ satisfies

$$\sum |\beta_2| \leq \mu TVX$$

where X is the associated configuration. Since the number of mesh strips in $[0,T]$ has order $1/\Delta x$ we conclude that the total strength of all such waves β_2 in $[0,T]$ satisfies

$$\sum |\beta_2| \leq c\mu/\Delta x \ .$$

This completes the proof of (11.10).

We note that optimal choice of μ, i.e. $\mu = (\Delta x/\sigma)^{1/2}$, leads to a growth estimate of the form

$$\underset{B_1 \cup B_2}{TV} \; u(\cdot,\cdot,\Delta x) \leq \text{const.}/(\Delta x \sigma)^{1/2} .$$

We conclude this section with a few brief remarks concerning the first term on the right hand side of (11.10). In this connection we shall first comment on the argument used in [26] to show that the error associated with the random choice method vanishes for an arbitrary equidistributed sequence. Fix $\varepsilon > 0$ and $T > 0$. In [26] the time strip $(0,T)$ is divided into substrips of equal length $k(\varepsilon)$ and the error associated with each substrip is estimated using a partitioning of elementary waves, cf. Lemma 3.2, [26]. It is then shown that in the limit as the mesh is refined the sum of the contributions of each strip has the order of ε. The partitioning divides the set of elementary waves of a given random choice difference approximation into two classes. The total magnitude of waves in the first class is small in the sense that their total contribution to the error has order ε independently of whether the sequence is equidistributed or not. The waves of the second class in a given substrip can be grouped into polygonal arcs Y in the x-t plane whose magnitude and speed of propagation are "nearly constant". This leads to an individual treatment of the arcs Y modelled on the argument used for a single-wave solution to a Riemann problem. Now if the equidistributed sequence admits a rate of approximation in the sense that the average number of its elements with index less than n in a given interval $J \subset (-1,1)$ satisfies

$$S(n,J)/n = |J| + c(J)\psi(n) \quad \text{and} \quad \lim_{n\to\infty} \psi(n) = 0$$

where the constant $c(J)$ is independent of n, then one may divide $(0,T)$ into substrips of equal length $k(\Delta x)$ satisfying

$$\lim_{\Delta x \to 0} k(\Delta x) = 0$$

and again add the contributions to each stip obtaining in the limit a total contribution on the order of ε provided that the length of the substrips approaches zero slower than the reciprocal of the error ψ, i.e. provided that

$$\lim_{\Delta x \to 0} \psi(1/\Delta x)/k(\Delta x) = 0 \ .$$

We recall that the hybridized schemes under discussion employ an equidistributed sequence and threshold function satisfying (11.1). Given such ψ and σ one can employ a similar two-class partitioning of elementary waves of the hybridized shceme to strips with length $\ell(\Delta x)$ satisfying

$$\lim_{\Delta x \to 0} \psi(1/\Delta x)/\ell(\Delta x) = 0 \quad \text{and} \quad \lim_{\Delta x \to 0} \ell(\Delta x)/(\sigma \Delta x)^{1/2} = 0$$

and show that the total contribution to the error from all waves of the first class plus all waves of the second class which form arc Y with length greater than say $\ell(\Delta x)/2$ has the order ε in the limit as the mesh is refined. Thus, one need only show that those segments of arcs Y with length less than $\ell(\Delta x)/2$ passing through diamonds in RC produce a total contribution which vanishes in the limit. To this end let us fix a substrip $S(\Delta x)$ of $(0,T)$ with length $\ell(\Delta x)$ and consider an arc Y of second class waves ε. The error associated with those segments of Y which lie in the set Ω of all RC diamonds in $S(\Delta x)$ has the order $\Delta x|\varepsilon|$. Without loss of generality we may assume that the magnitude of waves of Y remains a constant, denoted $|Y|$, since from the results of [26] the deviation from a constant leads to terms whose total sum vanishes in the limit. Hence the problem is to estimate

$$\Delta x \ r(Y)|Y|$$

where $r(Y)$ denotes the number of segments on Y which correspond to diamonds in Ω. Now, it follows from our previous analysis that we have an estimate of the form

$$\sum_{Y \in S} \tau(Y)|Y| \leq q\{S(\Delta x)\}/(\sigma\Delta x)^{1/2} ,$$

where $\tau(Y)$ denotes the number of times that the arc Y passes from Ω to Ω^c or from Ω^c to Ω and where the sum of the quantities q associated with each strip $S(\Delta x)$ is bounded independently of Δx for fixed T:

$$\sum \{q\{S(\Delta x)\} : S(\Delta x) \subset (0,T)\} \leq \text{const} .$$

Therefore, the contribution associated with one strip $S(\Delta x)$ satisfies

$$\sum_{Y \in S} \Delta x r(Y)|Y| \leq \sum_{Y \in S} \tau(Y)\ell(\Delta x)|Y| \leq \ell(\Delta x) q\{S(\Delta x)\}/(\sigma\Delta x)^{1/2} .$$

Summing over all strips $S(\Delta x)$ in $(0,T)$ yields

$$\sum_{S} \sum_{Y \in S} \Delta x r(Y)|Y| \leq \text{const.}\ell(\Delta x)/(\sigma\Delta x)^{1/2}$$

which implies that the total contribution *vanishes in the limit.*

12. THE ENTROPY CONDITION.

In this section we shall show that the limiting solutions u of our hybridized schemes satisfy Lax's entropy condition which requires that

(12.1) $$\eta(u)_t + q(u)_x \leq 0 ,$$

in the sense of distributions where η is a strictly convex entropy and q the associated entropy-flux [21]. Since the solutions u under consideration lie in $BV \cap L^\infty$ it is sufficient to show that

(12.2) $$\int_C \nu_t \eta(u) + \nu_x q(u) ds \leq 0$$

for a substantial class of contours C, e.g. parallelograms whose sides have slope $\pm\Delta t/\Delta x$. As a corollary of the entropy condition it follows that the entire sequence of associated difference approximations $u(\Delta x)$ converges to u as the mesh is refined in those circumstances where uniqueness is available

[9]; for example, if the initial data $u_0(x)$ gives rise to a piecewise Lipschitz solution u to a genuinely nonlinear system of two equations in the sense of [9] and if u satisfies (12.1), then u is unique within the class of $BV \cap L^\infty$ solutions satisfying (12.1). The class of piecewise Lipschitz solutions discussed in [9] contains in particular the classical solution of the Riemann problem.

Let us first consider an arbitrary conservative scheme in class K. A simple computation using the compatibility condition

$$\nabla\eta(u)\nabla f(u) = \nabla q(u)$$

shows that

$$\begin{aligned}
&\alpha_t \eta\{\theta(n,e)\} + \alpha_x q\{\theta(n,e)\} - \alpha_t \eta\{\theta(w,s)\} - \alpha_x q\{\theta(w,s)\} \\
&\quad + \beta_t \eta\{\psi(n,w)\} + \beta_x q\{\psi(n,w)\} - \beta_t \eta\{\psi(e,s)\} - \beta_x q\{\psi(e,s)\} \\
&\quad = O(|u_w - u_s|^2 + |u_s - u_e|^2)
\end{aligned}$$

Thus, the everywhere defined piecewise constant approximations $u = u(\Delta x)$ associated with a conservative scheme satisfy

$$\int_{\partial\Omega} \nu(u)ds = O(\Delta x)(|u_w - u_s|^2 + |u_s - u_e|^2)$$

for a typical mesh diamond Ω with values u_w, u_s, u_e at its west, south and east vertices. Thus a convergent sequence of difference approximations $u = u(\Delta x)$ which is stable in the total variation norm yields a solution u which satisfies (12.2) for all parallelograms C with sides of slope $\pm\Delta t/\Delta x$, provided that the maximum magnitude of waves in any bounded set Γ of the x-t plane approaches zero as the mesh is refined, i.e.

$$\lim_{\Delta x \to 0} \sup\{|u(x \pm \Delta x, t \pm \Delta t) - u(x,t)| : (x,t) \in \Gamma\} = 0$$

We note that exactly the same conclusion can be drawn for any scheme whose generating function is formed by a quadratic perturbation of a conservative

generator. Of course, in the absence of a condition on the limiting maximum wave magnitude, the problem of determining whether or not the entropy condition is satisfied is substantially more difficult.

For the original version of the random choice method, the entropy condition was verified by Lax [21]. For the deterministic verion involving an equidistributed sequence it follows immediately from the results of Liu [26] that the entropy condition is satisfied. In the latter case one has

$$\int_{\partial\Omega(i,j)} \nu(u)ds \leq E^{+}(i-1,j-1) + E^{-}(i+1,j-1)$$

where

$$E^{+} = \int_{0}^{\Delta x} \eta(u_n) - \eta(R(x/\Delta t;u_w,u_e))dx;\ E^{-} = \int_{-\Delta x}^{0} \eta(u_n) - \eta(R(x/\Delta t;u_w u_e))dx$$

where u_w and u_e denote the values of the corresponding grid function at the west and south vertices of the mesh diamond $\Omega(i,j)$. The inequality in (12.3) arises from the fact that the discontinuities of $u(\Delta x)$ in $\Omega(i,j)$ satisfy the entropy condition and the sum of all terms of the form $E^{\pm}$ vanish in the limit.

Therefore in view of our previous analysis, it follows that the limits of our hybridized scheme satisfy the entropy condition. We need only remark that the results of Section 11 show that the interior boundary contribution (associated with the set C and RC) to the contour integral

$$\int_{C} \nu_t \eta(u) + \nu_x q(u)ds\ ,$$

vanishes in the limit, since the quantity Δx times the total variation of $u(\Delta x)$ around said boundary vanishes in the limit.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study finite difference approximations to weak solutions of the Cauchy problem for hyperbolic systems of conservation laws in one space dimension. We establish stability in the total variation norm and convergence for a class of hybridized schemes which employ the random choice scheme together with perturbations of classical conservative schemes. We also establish partial stability results for classical conservative schemes. Our approach is based on an analysis of finite difference operators on local and global wave configurations.

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